The Use of Matrix Displacement Method for Vibrational Analysis of Structures

by
William O. Hughes[1]

Department of Mechanical Engineering and
Robotics Institute
Carnegie-Mellon University
Pittsburgh, PA 15213
May, 1981

ABSTRACT A study of the matrix displacement method for modeling the vibrations of structures is presented in this report. The model can analyze both the free and forced vibrations of a structure. Static loading on a structure is treated as a special case of the forced vibration analysis.

The work was supported in part by Carnegie-Mellon University Internal funding, through a Ford Foundation Research Grant, and by the Robotics Institute.

[1]Under the Supervision of Prof. W.L. Whittaker and Prof. A.J. Holzer

Table of Contents

1 Introduction 1
2 The Finite Element Method -- Fundamental Concepts and Applications 1
3 Explanation of the Model 2
3.1 Equations of Motion 2
3.2 The Matrix Displacement Method 4
3.3 Specific Aspects of Model 5
4 The Model: Examples and Accuracy 7
4.1 Example 1: Free Vibration of a Fixed-Free Uniform Beam 9
4.2 Example 2: Free Vibration of a Fixed-Fixed Uniform Beam 10
4.3 Example 3: Forced Vibration of a Fixed-Free Uniform Beam 10
4.4 Example 4: Static Deflection of a Fixed-Free Uniform Beam 13
4.5 Example 5: Static Deflection of a Fixed-Free Non-Uniform Beam 15
5 The Extension of the Model to Model A Turbine Blade 18
6 Conclusion 21

I. Appendix I Influence Coefficient Method 24

II. Appendix II Variational Method 26

III. Appendix III Computer Code of Model 31

List of Figures

Figure 1: The beam element and its forces, after Przemieniecki [7] 6

Figure 2: Stiffness Matrix of Beam Element of Figure 1 [After Przmieniecki]. [The sheer deformation parameters Φ_y and Φ can be considered to be zero.] 8

Figure 3: Consistent Mass Matrix for a Beam Element (After Przemieniecki [7]). 9

Figure 4: Example 1: Fixed-Free Uniform Beam. 9

Figure 5: First five bending mode shapes of Example 1. 12

Figure 6: First four axial mode shapes of Example 1. 14

Figure 7: Example 2: Fixed-Fixed Uniform Beam. 15

Figure 8: Example 3: Fixed-Free Uniform Beam With Dynamic Load. 15

Figure 9: Magnitude versus Forcing Frequency for Example 3. 16

Figure 10: Example 4: Fixed-Free Uniform Beam With Static Load. 17

Figure 11: Static Deflection of a Uniform Beam, Example 4 17

Figure 12: Example 5: Static Deflection of a Fixed-Free Non-Uniform Beam 18

Figure 13: Static Deflection of a Non-Uniform Beam, Example 5 19

Figure 14: Element Stiffness Influence Coefficients (After White, et al [10]). 24

Figure 15: Stiffness matrix of prismatic elements of Figure 14. 25

Figure 16: Axial element, cross-sectional area A, modulus E. 29

List of Tables

Table 1: Uniform Beam Properties . 8
Table 2: Natural frequencies (radians/sec) and Percentage Error (%) as a function of number of elements for Example 1. 11
Table 3: Calculated and Exact Natural Frequencies in Axial Mode. Calculated value used five element model, for Example 2. 13
Table 4: Calculated and Exact Values of Deflections for Example 4. 18

1 Introduction

A study of the matrix displacement method for modeling the vibrations of structures is presented in this report. The model can analyze both the free and forced vibrations of a structure. Static loading on a structure is treated as a special case of the forced vibration analysis.

A brief review of the Finite Element Method and its present use is first given. This is followed by a discussion of the methodology of the matrix displacement approach and a description of the specific model used. Examples of the use of the model to analyze the frequencies and mode shapes of the free and forced response of a beam structure and the static deflections of a beam structure are shown and compared with the closed form solutions. Finally, ways of extending the model to a more complicated structure, a turbine blade, are discussed. Conclusions are then drawn.

2 The Finite Element Method -- Fundamental Concepts and Applications

There are many methods available today which perform the analysis of structures. For example, in one method the structure is described by differential equations. The differential equations are then solved by analytical or numerical methods. Another method of analysis is the finite element method (FEM).

In this method, the structure is idealized into an assembly of discrete structural elements, each having an assumed form of displacement or stress distribution. The complete solution is then obtained by assembling these individual, approximate, displacement or stress distributions in a way satisfying the force equilibrium equations, the constitutive relationships of the material, the displacement compatibility between and within the elements and the boundary conditions of the structure.

Methods based on discrete element idealization have been used extensively in structural analysis.The early pioneering works of Turner, et al., in 1956 [1], and Argyris in 1960 [2] led to the application of this method to static and dynamic analysis of aircraft structures. Other fields of structural engineering, such as nuclear reactor design and ship construction have since employed this method.

Nor is the idea of discrete elements limited in use to structural analysis only. The fundamental concept of the finite element method is that any continuous quantity, such as displacements, temperature, or pressure, can be approximated by a finite number of elements. Thus, this approach can be used to solve problems in heat flow, fluid dynamics, electro-magnetics, fracture mechanics and seepage flow to name just a few other areas of usage.

The representation of a continuous structure by structural elements of finite size results in large systems of algebraic equations. A convenient way of handling these sets of equations is by the use of matrix algebra, which also has the advantage of being ideally suited for computations on high-speed digital computers. For this reason, expressions such as "matrix methods of structural analysis" are sometimes used to describe the method. More common though is the term "finite element method", which emphasizes the discretisation of the structure.

The finite element method actually encompasses three classes of matrix methods of structural analysis. The first is the displacement (or stiffness method), where the displacements of the nodes are considered the unknowns. The correct set of displacements results from satisfying the equations of force equilibrium. The second method is the force (or flexibility) method. Here the nodal forces are the unknowns and are found by satisfying the conditions of compatible of deformations of the members. The third class of matrix method is the mixed method, which is a combined force-displacement method.

One last comment on the finite element method in general is necessary. An error is introduced into the solution of the original problem as soon as the continuous structure is replaced by discrete elements. This error remains, even when the discrete element analysis is performed exactly. In general this error is reduced by increasing the number of discrete elements, thereby decreasing the element size and thus giving a better idealization of the continuous structure. Zienkiewicz, Brotton and Morton [3] suggest that the user may determine the limits of his error by: "(a) comparison of finite element calculations with exact solutions for cases similar to his specific problem; (b) a 'convergence study' in which two or more solutions are obtained using progessively finer subdivisions and the results plotted to establish their trend or (c) using experience of previous calculations as a guide to the treatment of the specific problem." Further information on matrix structural analysis and the finite element method may be found in many sources. [4-11]

3 Explanation of the Model

The following discussion is divided into three sections. Firstly the equations of motion will be stated. Secondly, the matrix displacement method for solving such equations will be described. Finally some specific aspects of the particular model being used will be discussed.

3.1 Equations of Motion

The motion of a vibrating system, consisting of mass and stiffness, of n degrees of freedom can be represented by n differential equations of motion. These equations of motion may be obtained by Newton's second law of motion, by Lagrange's equation or by the Influence Coefficients method. Since the equations of motion, in general, are not independent of each other, a simultaneous solution of these equations is required to calculate the frequencies of the system.

The matrix equation for the free vibration case is:

$$[K - \omega^2 M]\,[X] = [0] \tag{1}$$

where

[K]	represents the stiffness matrix of the structure,
[M]	represents the inertial (mass) matrix of the structure,
ω	represents the set of eigenvalues of the equations corresponding to the set of natural frequencies,
[X]	represents the set of eigenfunctions of the equations corresponding to the set of displacements

For the free vibration case the set of forces is just zero.

The matrix $[K-\omega^2 M]$ is called the impedance matrix.

The matrix equation for the forced vibration case is:

$$[K - \omega_f^2 \; M]\,[X] = [P] \tag{2}$$

where

[P]	represents the set of forces on the structure, and
ω_f	is the driving or forcing frequency.

The other terms are as previously defined.

Inspection of equations (1) and (2) reveals that neither contain damping terms. This is because structures of immediate concern have very low damping ($\sim 1 \times 10^{-4}$ critical damping).

An excellent treatment on the dynamics of structures is Clough and Penzien [14].

3.2 The Matrix Displacement Method

An outline of the application of the matrix displacement method in finite element analysis for the solution of dynamic problems follows. A similar outline is given by Zienkiewicz, et al. [3] for static analysis.

1. **Input**

   a. *Idealization of the problem*

   The continuous structure is divided into a number of elements. These elements are connected at common nodal points or nodes. It is at these nodes that the value of the continuous quantity (displacement) is to be determined.

   b. *Preparation of the data for the structure*

   The geometry of the structure is defined by assigning coordinates to the nodal points. The physical properties of the elements (dimensions, material parameters) are inputted.

   c. *Preparation of the load data*

   The loads to be applied to each element or node are defined.

   d. *Preparation of the boundary conditions or constraints*

   The prescribed constraints on the degrees of freedom and boundary conditions are stated.

2. **Processing**

   a. *Element Formulation*

   The stiffness and inertial matrices for each element are determined by the approximate relationships and the corresponding loads are calculated.

   b. *Assembly of the structure*

   The summation of the elemental matrices to form structural stiffness, inertial and load matrices is performed.

   c. *Reduction of equations*

   The boundary conditions and constraints in terms of certain specified displacements are introduced, thereby reducing the number of equations to be solved.

   d. *Solution of simultaneous equations*

   The solution of the eigen problem of equation (1) or (2) results in the natural frequencies of the structure (eigenvalues) and the modal shapes or displacements of the nodes (eigenfunctions).

c. *Calculation of stresses*

If required, the elemental stresses could be calculated from the nodal displacements and elemental stiffness.

3. **Output**

The results of the solution to the eigenvalue problem and the stress calculation are presented in an easily interpreted form.

3.3 Specific Aspects of Model

This section is concerned with specific aspects of the model. The element and its formation will be discussed first. Information concerning the computer code and its subroutines will then be given.

1. Element Formulation

The element chosen for the model is the beam element which is given by Przemiemiecki [7]. This element was chosen so as to allow direct comparison of results with known solutions (see section 4). The beam element is a two node element. The model allows the nodes to have either three degrees of freedom (x and y, translational and rotation about *z*, i.e. motion confined to a plane) or six degrees of freedom (x,y,z translational, rotation about x,y,z, i.e. the general case).

Fig. 1 shows the beam element. The following forces act on the beam:

- axial forces s_1 and s_7
- shearing forces s_2, s_3, s_8, and s_9
- bending moments s_5, s_6, s_{11}, and s_{12}
- and twisting moments (torques) s_4 and s_{10}.

The location and positive directions of these forces are also given in Fig. 1. The corresponding displacements U_1, U_2,. . . U_{12} will be taken to be positive in the positive direction of these forces.

Each element has its own set of physical parameters. For the beam element these parameters are: Young's modules, cross-sectional area, moment of inertia about the *y* and *z* axis, Poisson's ratio, mass density, and length (along *x* axis). All of these parameters are inputted directly except for the length which is computed from the inputted coordinates of the nodes.

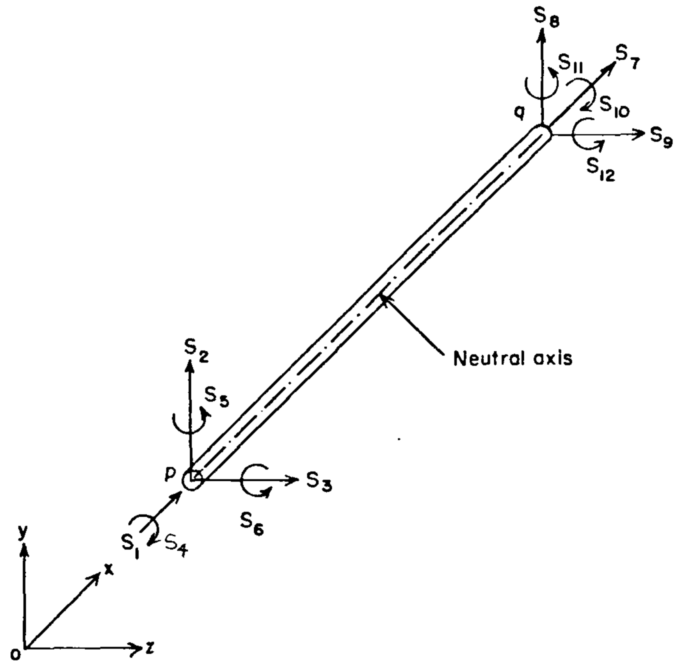

Figure 1: The beam element and its forces, after Przemieniecki [7].

The model performs calculations for either the free or forced vibration case. To perform such calculations requires the calculation of the structural stiffness and inertial matrices, along with information of the loading and boundary conditions of the structure. The effect of constraining a degree of freedom is to strike out the corresponding rows and columns of the stiffness, interial and load matrices.

The stiffness matrix for a beam element is shown in Fig. 2. The shear deformation parameters Φ_y and Φ_z can be taken as zero. This matrix may be obtained in various ways, two of which are the influence coefficients method and the variational method, which are outlined in Appendices I and II.

The inertial matrix for the beam element is shown in Fig.3. This matrix is obtained by the same methods as the stiffness matrix, as described in Appendices I and II.

Liepens [13] gives a third way of calculating the stiffness and inertial matrices.

The structural matrix for both stiffness and inertia is obtained by superposition of the individual elemental matrices. Actual superposition occurs only when degrees of freedom are common to more than one element.

2. Computer Coding

The computer code itself contains ten subroutines, called by the main program, entitled VIBRAT. A brief explanation of the subroutines will now be given.

INPUT - This subroutine asks the user for the necessary information which is needed to assemble the structure. Information such as: free or forced case, number of elements, coordinates of nodes, physical parameters, structural loading, and constrained degrees of freedom are inputted in this section.

CONECT - This subroutine establishes the geometry of the model. It determines the distances between adjacent nodes of the structure.

KMAT - This subroutine calculates the elemental stiffness matrix for each element and then assembles the structural stiffness matrix from them.

MMAT - This is similar to KMAT only here the mass or inertial matrices are calculated.

EIGEN - This subroutine is called for the free vibration case. The purpose of it is to calculate the eigenvalues (natural frequencies) and eigenvectors (mode shapes) of equation (1). This subroutine calls two other subroutines: EIGZF, an IMSL routine which actually does the solving, and CLAMPR, which determines which degrees of freedom are constrained.

SOLVE - This subroutine is called for the forced vibration case. This routine solves equation (2) for the displacement. This subroutine also calls two other subroutines: LEQT1F, an IMSL routine which does the solving, and CLAMPR, which determines the proper degrees of freedom to be constrained.

REMARK - is a subroutine whose purpose is to explain the use of the main program VIBRAT and its subroutines. Information on the nomenclature and file structure used can be found in REMARK. The user of the model is recommended to refer to REMARK if he has any questions on the computer code used in this model.

The code for all of these routines may be found in Appendix III.

4 The Model: Examples and Accuracy

This section presents various examples of use of the model. The examples chosen represent five types of possible problems. They are:

1. free vibration of a fixed-free uniform beam

2. free vibration of a fixed-fixed uniform beam

3. forced vibration of a fixed-free uniform beam

4. static deflection of a fixed-free uniform beam

5. static deflection of a fixed-free non-uniform beam.

The accuracy of each example is discussed.

$$
\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \\ S_{11} \\ S_{12} \end{bmatrix}
=
\begin{bmatrix}
\frac{EA}{l} & & & & & & & & & & & \\
0 & \frac{12EI_z}{l^3(1+\Phi_y)} & & & & & & & & & \text{Symmetric} & \\
0 & 0 & \frac{12EI_y}{l^3(1+\Phi_z)} & & & & & & & & & \\
0 & 0 & 0 & \frac{GJ}{l} & & & & & & & & \\
0 & 0 & \frac{-6EI_y}{l^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{l(1+\Phi_z)} & & & & & & & \\
0 & \frac{6EI_z}{l^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{l(1+\Phi_y)} & & & & & & \\
\frac{-EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{l} & & & & & \\
0 & \frac{-12EI_z}{l^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{l^2(1+\Phi_y)} & 0 & \frac{12EI_z}{l^3(1+\Phi_y)} & & & & \\
0 & 0 & \frac{-12EI_y}{l^3(1+\Phi_z)} & 0 & \frac{6EI_y}{l^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_y}{l^3(1+\Phi_z)} & & & \\
0 & 0 & 0 & \frac{-GJ}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{l} & & \\
0 & 0 & \frac{-6EI_y}{l^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{l(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{l^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{l(1+\Phi_z)} & \\
0 & \frac{6EI_z}{l^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{l(1+\Phi_y)} & 0 & \frac{-6EI_z}{l^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{l(1+\Phi_y)}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
$$

Figure 2: Stiffness Matrix of Beam Element of Figure 1 [After Przmieniecki]. [The sheer deformation parameters Φ_y and Φ can be considered to be zero.]

The first four examples use the geometric and material values listed in Table 1.

Parameter	Value	Units
Total Beam Length (L)	25.0	inches
Young's Modulus (E)	27.8 x 10^6	pounds force/inches^2
Cross-Sectional Area (A)	2.0	inches^2
Moment of Inertia about Z-Axis (I_z)	0.2	inches^4
Moment of Inertia about Y-Axis (I_y)	0.7	inches^4
Poisson's Ratio (ν)	0.305	-----
Mass Density (ρ)	0.283	pounds mass/inches^3

Table 1: Uniform Beam Properties.

$\mathbf{m} = \rho A l$ ×

	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{3}$											Symmetric
2	0	$\frac{13}{35}+\frac{6I_z}{5Al^2}$										
3	0	0	$\frac{13}{35}+\frac{6I_y}{5Al^2}$									
4	0	0	0	$\frac{J_x}{3A}$								
5	0	0	$-\frac{11l}{210}-\frac{I_y}{10Al}$	0	$\frac{l^2}{105}+\frac{2I_y}{15A}$							
6	0	$\frac{11l}{210}+\frac{I_z}{10Al}$	0	0	0	$\frac{l^2}{105}+\frac{2I_z}{15A}$						
7	$\frac{1}{6}$	0	0	0	0	0	$\frac{1}{3}$					
8	0	$\frac{9}{70}-\frac{6I_z}{5Al^2}$	0	0	0	$\frac{13l}{420}-\frac{I_z}{10Al}$	0	$\frac{13}{35}+\frac{6I_z}{5Al^2}$				
9	0	0	$\frac{9}{70}-\frac{6I_y}{5Al^2}$	0	$-\frac{13l}{420}+\frac{I_y}{10Al}$	0	0	0	$\frac{13}{35}+\frac{6I_y}{5Al^2}$			
10	0	0	0	$\frac{J_x}{6A}$	0	0	0	0	0	$\frac{J_x}{3A}$		
11	0	0	$\frac{13l}{420}-\frac{I_y}{10Al}$	0	$-\frac{l^2}{140}-\frac{I_y}{30A}$	0	0	0	$\frac{11l}{210}+\frac{I_y}{10Al}$	0	$\frac{l^2}{105}+\frac{2I_y}{15A}$	
12	0	$-\frac{13l}{420}+\frac{I_z}{10Al}$	0	0	0	$-\frac{l^2}{140}-\frac{I_z}{30A}$	0	$-\frac{11l}{210}-\frac{I_z}{10Al}$	0	0	0	$\frac{l^2}{105}+\frac{2I_z}{15A}$

Figure 3: Consistent Mass Matrix for a Beam Element (After Przemieniecki [7]).

4.1 Example 1: Free Vibration of a Fixed-Free Uniform Beam

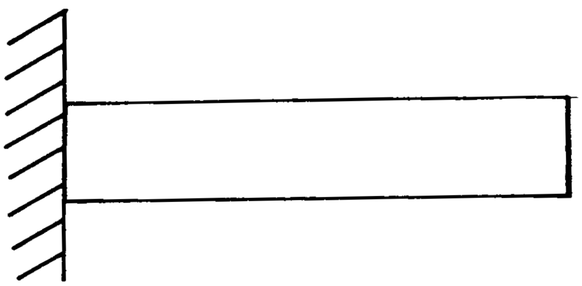

Figure 4: Example 1: Fixed-Free Uniform Beam.

Table 2 summarizes the results for this problem, using one, two, and five elements. It is clear that increasing the number of elements increases the accuracy of the results, and this supports the statements of Zienkiewicz given earlier.

The natural frequencies calculated by the model are compared with the closed form solution obtained from the partial differential equation of the continuous system. For the fixed-free case the closed form solutions are:

Axial $$\omega = \frac{n\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad \text{where } n = 1, 3, 5, \ldots \tag{3}$$

Bending(i) $$\omega = a^2L^2\sqrt{\frac{EI_i}{\rho AL4}} \qquad \text{where } 1 + \cos aL \cosh aL = 0 \qquad i = Y \text{ or } Z \tag{4}$$

Torsional $$\omega = \frac{n\pi}{2L}\sqrt{\frac{G}{\rho}} \qquad \text{where } n = 1, 3, 5, \ldots G = \frac{E}{2(1+\nu)} \tag{5}$$

Thus from Table 2, one can see that by using just five elements, the model gives ten transverse modes, two axial modes, and two rotational modes, the frequencies of which are all within 5% of the exact solutions. Again, clearly greater accuracy of results and more (higher) modes may be accomplished by increasing the number of elements.

Diagrams of the mode shapes for the first five bending modes (in Y) and the first four axial modes (along X) are given in Figs. 5 and 6. The model shapes agree with the closed form predictions in every case.

4.2 Example 2: Free Vibration of a Fixed-Fixed Uniform Beam

In this example the beam is held fixed on both ends. See Figure 7 . Table 3 shows the calculated and exact values for the axial mode natural frequencies. The accuracy is similar to that of example 1.

4.3 Example 3: Forced Vibration of a Fixed-Free Uniform Beam

In this example (Figure 8), the beam is subjected to a harmonically varying load P(t) of amplitude P and circular frequency, ω_f. Figure 9 is a plot of the magnitude in the transverse direction of the free end node. As expected, as ω_f approaches a natural frequency (those found in example 1), a resonance condition occurs resulting in very large magnitudes of deflection. The expression for the amplitude of response A is given by

Number of Elements	Axial Mode Frequencies	Bending (in y) Mode Frequencies	Bending (in z) Mode Frequencies	Torsional Mode Frequencies
1	13,491 (10.3)	348 (0.6)	650 (0.3)	8,350 (10.3)
		3,413 (57.2)	6,313 (55.5)	
2	12,551 (2.6)	347 (0.3)	647 (0.2)	7,769 (2.6)
	43,847 (19.5)	2,183 (0.6)	4,058 (0.1)	27,140 (19.5)
		7,350 (20.9)	13,509 (18.8)	
		20,959 (75.9)	37,022 (66.1)	
5	12,285 (0.4)	344 (0.6)	641 (1.1)	7,604 (0.4)
	38,074 (3.7)	2,166 (0.2)	4,027 (0.8)	23,567 (3.7)
	67,455 (10.3)	6,063 (0.3)	11,173 (1.8)	41,753 (10.3)
	101,152 (18.1)	11,914 (0.0)	21,684 (2.7)	62,611 (18.1)
	130,102 (18.2)	19,644 (0.2)	35,225 (4.4)	80,531 (18.2)
Exact	12,235	346	648	7,573
	36,704	2,171	4,061	22,719
	61,170	6,079	11,373	37,865
	85,645	11,912	22,285	53,011
	110,115	19,693	36,843	68,157

Table 2: Natural frequencies (radians/sec) and Percentage Error (%) as a function of number of elements for Example 1.

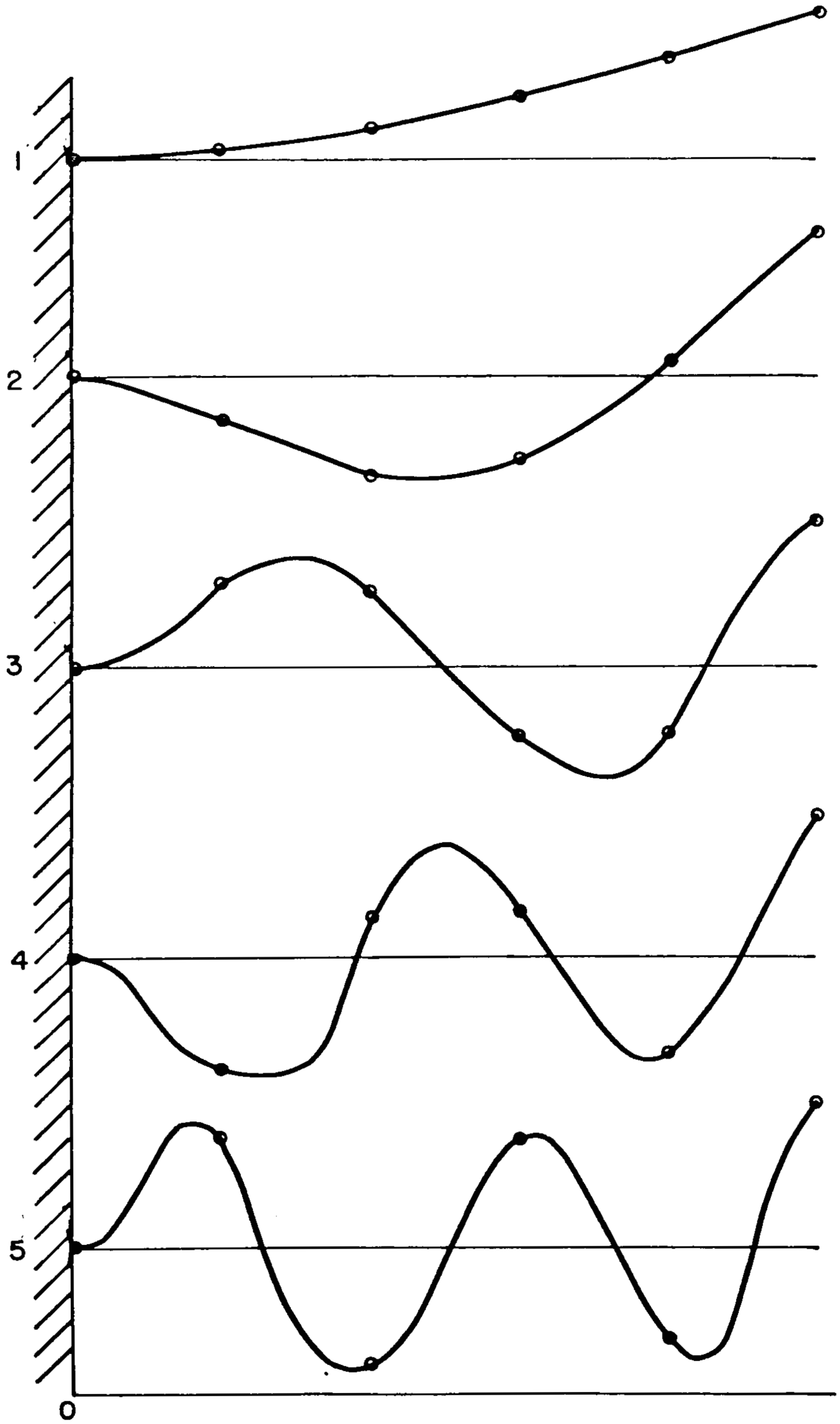

Figure 5: First five bending mode shapes of Example 1

Axial Mode	Calculated Natural Frequency (rad/sec)	Exact Natural Frequency (rad/sec)	% Error
1	24,874	24,470	1.7
2	52,186	48,940	6.6
3	83,933	73,410	14.3
4	117,570	97,880	20.1

Table 3: Calculated and Exact Natural Frequencies in Axial Mode. Calculated value used five element model, for Example 2.

$$A = \frac{P_o}{K(1-\beta^2)} = \frac{P_o}{K} D \qquad (6)$$

where Po/K represents the static deflection,
β equals the ratio of the forcing frequency to natural frequency,
D dynamic magnification factor equal to $1/(1-\beta^2)$

Analysis of the calculated amplitude in terms of the dynamic magnification factor agrees with equation (6) in those frequency regions dominated by just one natural frequency.

4.4 Example 4: Static Deflection of a Fixed-Free Uniform Beam

By letting the driving frequency, ω_f, be zero in the forced vibration option, the model is able to solve static deflection problems. Figure 11 shows the deflection of the beam under the static loading of example 4. The model's calculations, using just five elements are within 2% of the exact beam theory results. The deflection and slope at the end of the beam are given by the expressions:

$$\Delta = PL^3/3EI$$
$$\Theta = PL^2/2EI$$

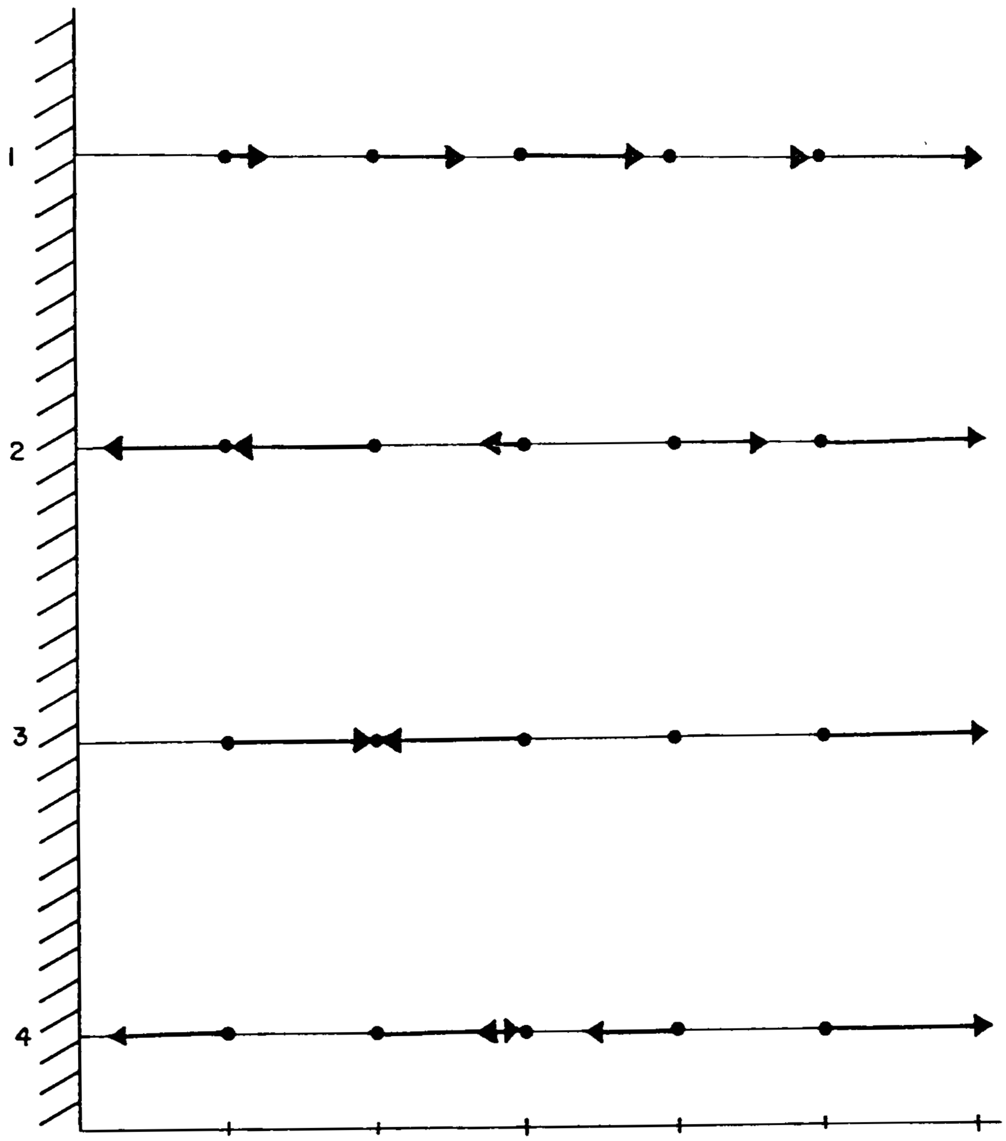

Figure 6: First four axial mode shapes of Example 1.

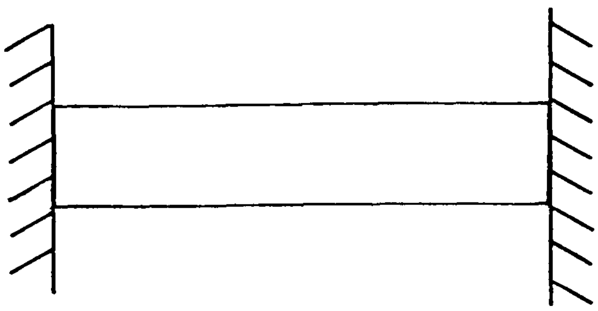

Figure 7: Example 2: Fixed-Fixed Uniform Beam.

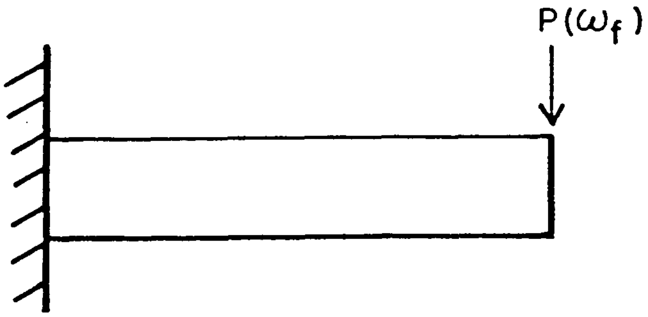

Figure 8: Example 3: Fixed-Free Uniform Beam With Dynamic Load.

Values calculated using these expressions are compared with the model results in Table 4.

4.5 Example 5: Static Deflection of a Fixed-Free Non-Uniform Beam

Until now, all the examples have dealt with uniform beams. Example 5 is an example taken from Laursen [11]. Laursen solves the problem in three differential ways: by the moment-area method, by the conjugate beam method, and by Newmark's method. The solution for displacement and slope at the free end is given as:

$$\Delta = -0.457 \text{ inches}$$

$$\Theta = -0.0041 \text{ radians}$$

The model gives identical results.

A sketch of the deflection is shown in Figure 13.

The purpose of the previous five examples is to illustrate the use and application of the model to a variety

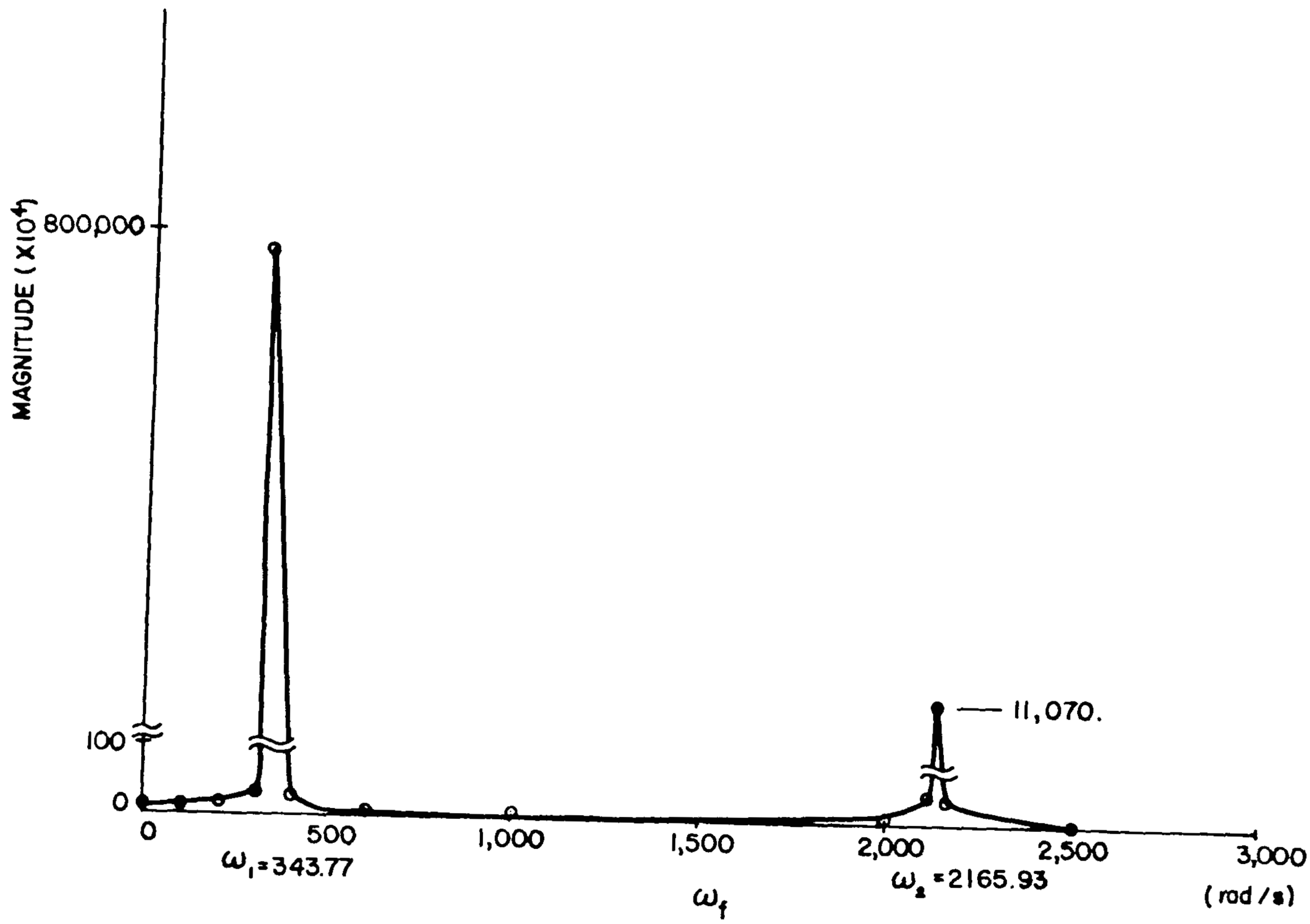

Figure 9: Magnitude versus Forcing Frequency for Example 3.

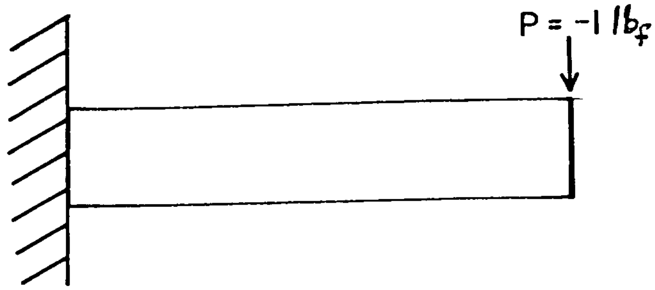

Figure 10: Example 4: Fixed-Free Uniform Beam With Static Load.

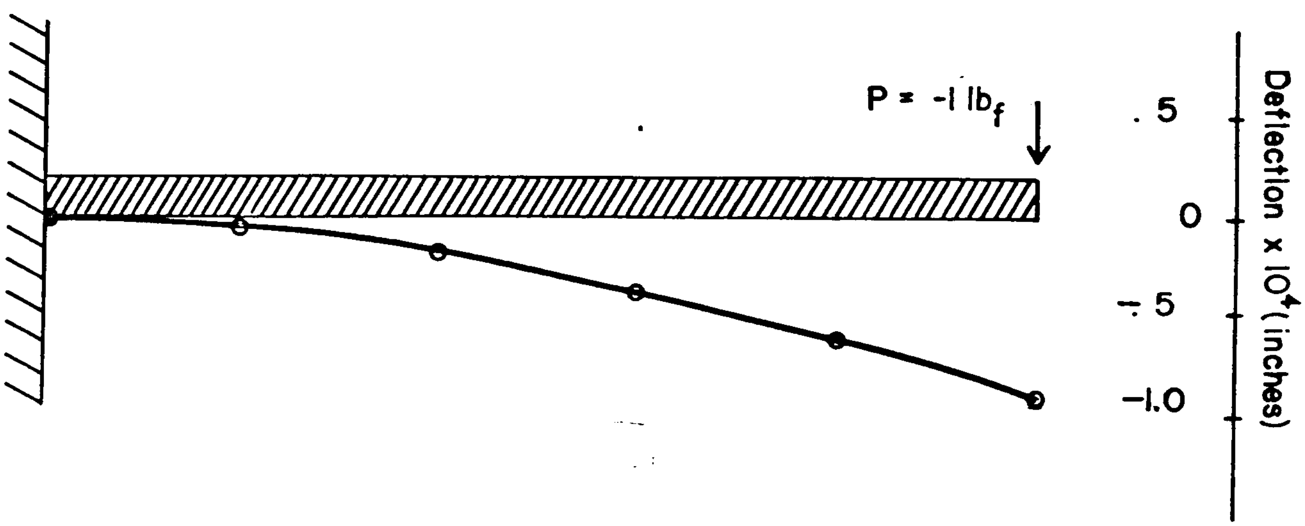

Figure 11: Static Deflection of a Uniform Beam, Example 4.

of cases. Other cases of a more complicated nature could have been solved as easily, however these examples give the user some insight into the accuracy of the solution obtained. They also indicate that very accurate results are obtained by the model with relatively few elements. In general, for a more complicated structure more elements will be required to obtain an accurate model. Techniques for handling more complex structures are discussed in the next section.

	Δ (inches)	θ (radians)
Exact	-9.37×10^{-4}	-5.62×10^{-5}
Calculated	-9.50×10^{-4}	-5.69×10^{-5}
%	1.4	1.2

Table 4: Calculated and Exact Values of Deflections for Example 4.

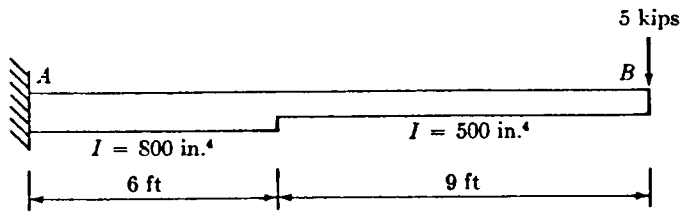

Figure 12: Example 5: Static Deflection of a Fixed-Free Non-Uniform Beam. [After Laursen].

5 The Extension of the Model to Model A Turbine Blade

An example of a more complicated structure which might be of vibrational interest to an engineer is a turbine blade. The equations of motion for a beam in bending vibration is a fourth-order differential equation, whose solution is easily found. The solution for a non-uniform and asymmetrical beam is much more complicated. A tapered, pre-twisted turbine blade with airfoil cross-section might be modeled as such a beam.

The differential equations for combined flapwise bending, chordwise bending and torsion of a twisted non-uniform blade are derived by Houbolt and Brooks [16]. The solutions of these equations for the continuous system have not been found. Thus the analysis of such structures are limited to special cases

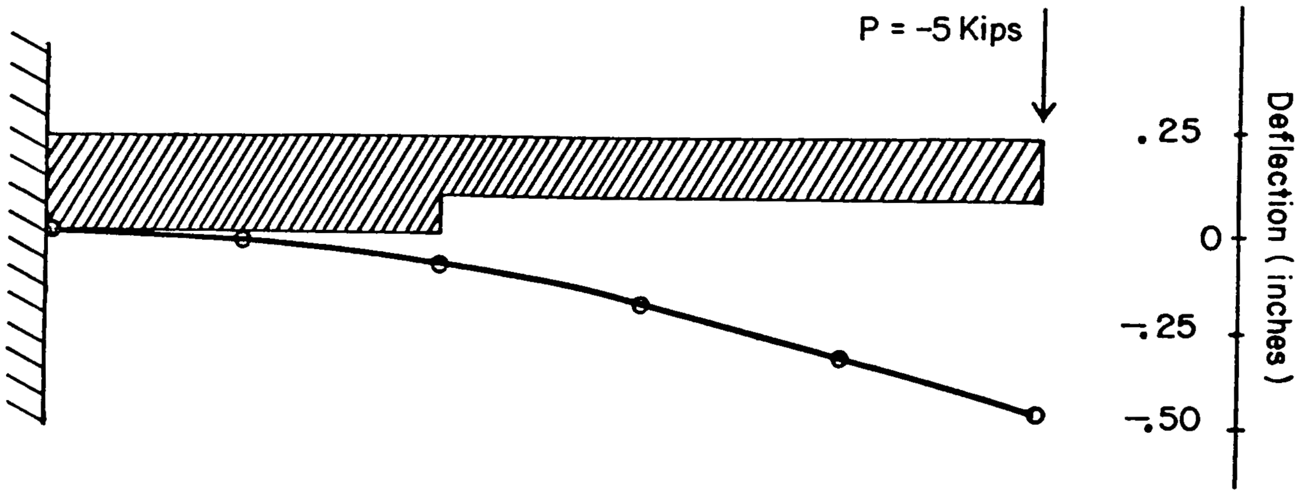

Figure 13: Static Deflection of a Non-Uniform Beam, Example 5.

which solutions are obtainable, or to approximate solutions. Various techniques of an analytical and iterative nature such as the Myklestad method, Holzer method, Stodala method, Rayleigh-Ritz method, transmission matrix method, and the Runge-Kutta method have been studied [14]. A few typical examples are given in the references [15,17-20].

The application of the model presented in this report to the turbine blade would be a very useful tool to the engineer and his study of the blade's free and forced vibrations.

The model allows each element to have its own set of geometric and physical parameters. Thus neither the non-uniformity or tapering of the blade would lead to any modeling problems. However the airfoil shape of the blade would not have the same torsional stiffness as a beam. Thus the first adaptation to the model needed would be to correctly compute the torsional stiffness for an airfoil shape and input this into the model rather than using that which the model computes.

There is another problem which arises from the twisting and geometry of the turbine blade. The natural frequencies of such a blade are coupled frequencies with the mode shapes consisting in general of transverse motion coupled with torsion. The coupling is dependent upon the degree of pre-twist and the ratio of depth taper to width taper. For a given blade, coupling becomes stronger with increasing pre-twist and with increasing width to depth taper ratio.

The simulation of this coupling in the model could be accomplished by either introducing it through the element itself or through the geometry of the structure. The first way implies changing the element from a beam element to a new element. This new element could be derived from a variational method (see Appendix II) applied to the differential equations for the blade equations derived by Houbolt and Brooks [16]. The ideal of coupling through the geometry of the structure implies the use of additional beam elements. Part of these elements would be used to form the center of stiffness for the blade which would now be a curve rather than the straight line used thus far. Other elements could extend at right angles from this curve. These elements would act primarily as lumped masses and form the curve representing the center of mass of the blade.

Modeling a turbine blade with this model would require some additional work to implement the ideas presented in this section. However the matrix displacement method used is a very powerful one and the use of the model and extensions of it are applicable to a wide range of problems in vibrational analysis of structures. Building a library of elements would greatly extend the usefulness of the existing model, and additionally, the introduction of element rotation would lead to further improvement.

6 Conclusion

This report primarily concerns itself with three topics:

1. the explanation of the matrix displacement method for use in vibrational analysis of structures,

2. specific examples showing the variety and accuracy of the method, and

3. possible extensions of the model to allow for application to an even wider variety of problems.

The model presented here currently allows for only one type of element, the beam element. It has been shown that by using just a few beam elements very accurate results of frequencies and modal shape are obtained for beam-like structures. Creating a library of element types would allow the user even greater flexibility. The accuracy of the model using these new elements should be comparable to that presented here.

Acknowledgements

The author is grateful to Prof. Alex J. Holzer, who acted as advisor throughout this study. Special thanks also goes to Prof. William L. Whittaker of the Department of Civil Engineering at Carnegie-Mellon University, for the help, guidance and encouragement given.

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I. Appendix I Influence Coefficient Method

One method of obtaining the stiffness matrix is the influence coefficient method. This method is widely used in structural analysis with static loadings [10,11]. There are both stiffness and flexibility influence coefficients : only the stiffness influence coefficients will be considered here.

The stiffness coefficients for an element are found by alternatively constraining all degrees of freedom but one and displacing this one by a unit amount. The resulting forces on the other degrees of freedom are the stiffness coefficients. That is K_{ij} is the force or couple corresponding to degree of freedom i due to the unit displacement of degree of freedom j. In Fig. 14 a prismatic element of length l, area A, moment of inertia about the Z axis I, and modulus of elasticity E, with three degrees of freedom per node is shown.

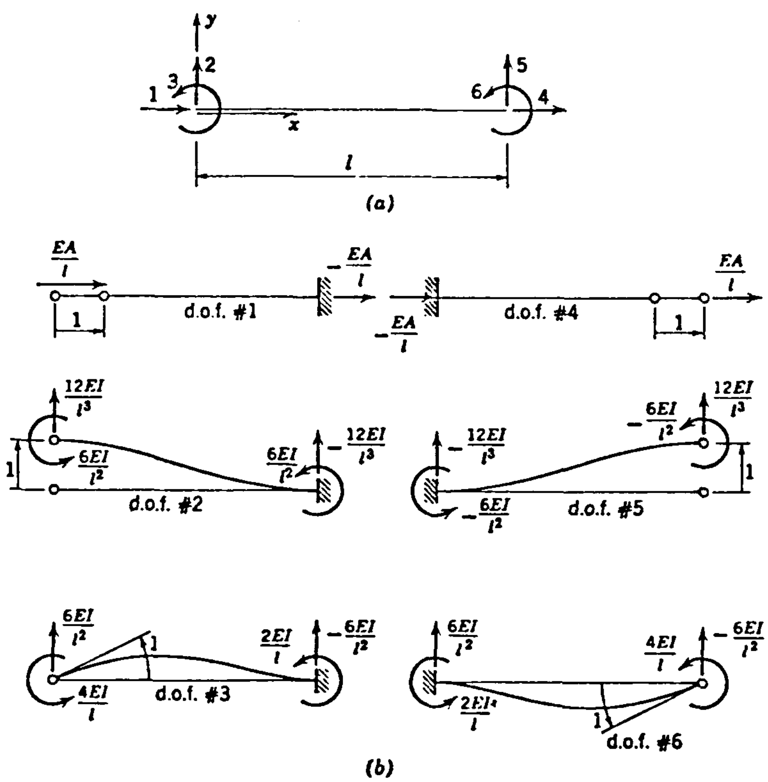

Figure 14: Element Stiffness Influence Coefficients (After White, et al [10]).

By performing the stiffness influence method procedure on this element, the stiffness matrix is obtained:

$$[k'] = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & \frac{-EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & \frac{-12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & \frac{-6EI}{l^2} & \frac{2EI}{l} \\ \frac{-EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & \frac{-12EI}{l^3} & \frac{-6EI}{l^2} & 0 & \frac{12EI}{l^3} & \frac{-6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & \frac{-6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

Figure 15: Stiffness matrix of prismatic elements of Figure 14.

Comparison of Fig. 2 and 15 shows that the matrix of Figure 15 is contained within the matrix of Figure 2. In Fig. 15, each node has three degrees of freedom, in Fig. 2 there are six degrees of freedom per node.

The inertial (or mass) matrix may be calculated similarly. The mass influence coefficients would represent the mass inertia force acting at a degree of freedom due to a unit acceleration of another degree of freedom.

II. Appendix II Variational Method

Another method of computing elemental stiffness matrices is the variational or energy method commonly used in finite element programs. The outline presented here largely follows that of Gallagher [8].

The principle of minimum potential energy furnishes a variational basis for the formulation of the element stiffness matrix. The potential energy (π_p) of a structure is given by the strain energy (U) plus the potential of the external work V ($V = -W_{ext}$). The theorem of potential energy is: of all displacements, satisfying the boundary conditions, those that satisfy the equilibrium conditions make the potential energy assume a stationary (extreme) value. Thus

$$\pi_p = U + V \tag{7}$$

$$\delta\pi_p = \delta U + \delta V = 0 \tag{8}$$

And for stable equilibrium, π_p is a minimum.

$$\delta^2\pi_p = \delta^2 U + \delta^2 V > 0 \tag{9}$$

The change in strain energy density due to the change in strain caused by a virtual displacement ($\delta\varepsilon$) is given by

$$\delta\,(dU) = \sigma\,\delta\varepsilon \tag{10}$$

Where σ is the equilibrium stress state prior to the application of the virtual displacement. The stress--strain law is

$$\sigma = [E]\varepsilon - [E]\varepsilon^{int} \tag{11}$$

where [E] is called the material stiffness matrix, a matrix of elastic constants. For simplicity, let there be no initial strain. Substitution of (11) into (10) yields

$$\delta(dU) = \varepsilon\,[E]\delta\varepsilon \tag{12}$$

Integration between zero and the strain ε, corresponding to σ, gives

$$dU = \frac{1}{2}\varepsilon\,[E]\varepsilon \tag{13}$$

and integration over the volume of the element results in

$$U = \frac{1}{2}\int_{vol} \varepsilon[E]\varepsilon \; d(vol) \tag{14}$$

The variation of U is

$$\delta U = \int_{vol} \varepsilon[E]\,\delta\varepsilon \; d(vol) \tag{15}$$

The potential of the applied loads is

$$V = -\sum_{\iota=1} F_\iota \Delta_\iota - \int_{s_\sigma} \bar{T}\cdot \mathrm{U}ds \tag{16}$$

where F_ι represents point forces, and T are traction forces on the surface. The variation of V is

$$\delta V = -\Sigma F_\iota \delta\Delta_\iota - \int_{s_\sigma} \bar{T}\cdot \delta \mathrm{U}ds \tag{17}$$

Using the minimum potential energy theorem (equation 8) results in

$$\int_{vol} \varepsilon[E]\delta\varepsilon \; d(vol) + -\Sigma F_\iota \delta\Delta_\iota - \int_{s_\sigma} \bar{T}\cdot \delta \mathrm{U}ds = 0 \tag{18}$$

In the finite element matrix, the displacements, [Δ], are written as a polynomial matrix times a vector of parameters in the assumed displacement field.

$$[\Delta] = [P]\,[a] \tag{19}$$

[P] evaluated at the node gives a matrix [B], consisting of constants. Thus

$$[\Delta_{nodes}] = [B]\,[a] \tag{20}$$

Inverting to find [a] in (20) and substitution into (19) leads to

$$[\Delta] = [P]\,[B^{-1}]\,[\Delta nodes]$$

$$= [N]\,\Delta nodes \tag{21}$$

where N is the shape function. The shape function N_ι has the quality that it is equal to 1 when evaluated at the geometric coordinates of the point at which Δ_ι is defined and is equal to zero at all other degrees-of-freedom Δ_ι, $j \neq \iota$.

The matrix [D] is called the dof-to-strain transformation. Then

$$[\varepsilon] = [D]\,[\Delta nodes] \tag{22}$$

For example if,

$$\varepsilon = \frac{\partial u}{\partial x}, \text{ then}$$

$$[D] = [N'\,] \tag{23}$$

Substitution of these ideas into (18) leads to

$$\int_{vol} [D]^t \; [E][D]\Delta_{nodes} dVol(\delta\Delta nodes^t) - \Sigma[N_\iota]^t F\iota(\delta\Delta nodes^t)$$

$$-\int_s [N]^t[T]ds(\delta\Delta nodes^t) = 0 \tag{24}$$

dividing (24) by $\delta\Delta nodes^t$ results in

$$[K]\,\Delta nodes - Fext = 0 \tag{25}$$

where

$$[K] = \int_{vol} [D]^t [E]\,[D]\,dvol \tag{26}$$

$$Fext = \int_s [N]^t [T]\,dS + \sum [N_i]^t F_i \tag{27}$$

Thus the stiffness matrix can be found by equation (26).

As an example take the axial element show in Figure 16, with dof$\Delta 1$ and $\Delta 2$ only. The procedure to calculate the stiffness of this element follows. Let

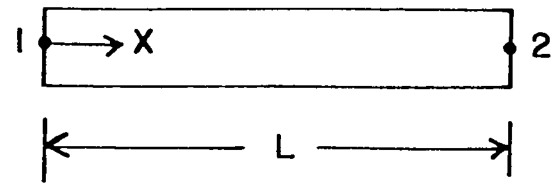

Figure 16: Axial element, cross-sectional area A, modulus E.

$$[\Delta] = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{bmatrix} 1 & x \end{bmatrix} \frac{1}{L} \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \Delta_{nodes} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \Delta_{nodes}$$

$$[D] = [N'] = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}$$

$$[K] = \int_{vol} \begin{bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{bmatrix} [E] \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} dVol$$

$$= EA \int_0^L \begin{bmatrix} \frac{1}{L^2} & \frac{-1}{L^2} \\ \frac{-1}{L^2} & \frac{1}{L^2} \end{bmatrix} dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The result is also contained in the stiffness matrices shown in Figures 2 and 15.

The inertial (or mass) matrix can also be calculated by use of this method. The variational approach leads to

$$[M] = \int_{vol} [N][\rho][N]^T dVol \qquad (28)$$

where [ρ] is the material mass density matrix. Since the shape functions used here are the same as those used for the stiffness calculation the result is called the consistent mass matrix. A consistent mass matrix is more accurate than a lumped mass approach [12].

III. Appendix III Computer Code of Model

Available from Author.