Computational Symmetry[a]

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[a] A modified version of this report appears as a book chapter in *Symmetry 2000*, Portland Press

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ABSTRACT

This paper gives a sampler on an emerging area of research and applications, namely, *computational symmetry*. The introduction of computers poses challenging tasks for machine representation and reasoning about symmetry and group theory. We demonstrate, through three concrete applications, the power, the difficulties and feasibility of using symmetry and group theory on computers. A computational framework is proposed to study symmetry in a multi-dimensional, continuous space.

1 Introduction

Symmetry is pervasive in both natural and man-made environments [5, 10, 9, 31, 32, 34, 35]. Humans have an innate ability to perceive and take advantage of symmetry [13] in everyday life, but it is not obvious how to automate this powerful insight. The introduction of computers poses challenges for machine representation and reasoning about symmetry. It is a continuous effort of the author to develop computational tools for dealing with symmetry in various applications using computers.

2 Computational Symmetry and Its Applications

Computational symmetry refers to the practice of representing, detecting, and reasoning about symmetries on computers. The reasons to care about computational symmetry in computer science are many-fold:

- Symmetry *exists* everywhere;
- symmetry is intellectually *stimulating*;
- symmetry implies a structure, that can be either *helpful* or *harmful* in applications;
- machine computation of symmetry is *challenging*: it has to connect abstract mathematics with the noisy, imperfect, real world;
- few computational tools exist for dealing with real world symmetries.

We demonstrate, through three concrete applications, the power, the difficulties and the feasibility of using symmetry on computers. These applications are a robot assembly planner, an intelligent neural-radiology image database, and a computational model for periodic pattern perception.

2.1 A Group Theoretical Formalization of Surface Contact

One basic question in robotics automation is **how to describe contacts between solids to a robot**? For example, how would you ask a robot to put a cube in a corner? This seemingly simple task requires 24 equivalent, but different, sets of task specifications if you wish to enumerate all the geometric possibilities. It is a non-trivial task to communicate the *full* range of spatial relationships between locally symmetrical objects with a robot that does not *understand* symmetry. Such task specifications are forced to be either tedious and redundant, or suffering from incompleteness.

Current engineering practice is still limited to a finite set of case-based scenarios. Computing the relative positions of solids that are in contact is a fundamental problem in many fields, including robotics, computer graphics, computer aided design and manufacturing (CAD & CAM) and computer vision. It is the focus of this work to formalize solid contact based on local symmetry, to construct a computational framework using group theory, and to demonstrate the effectiveness of applications of computational group theory in robotics [15, 23]. It can be shown (Figure 1) that

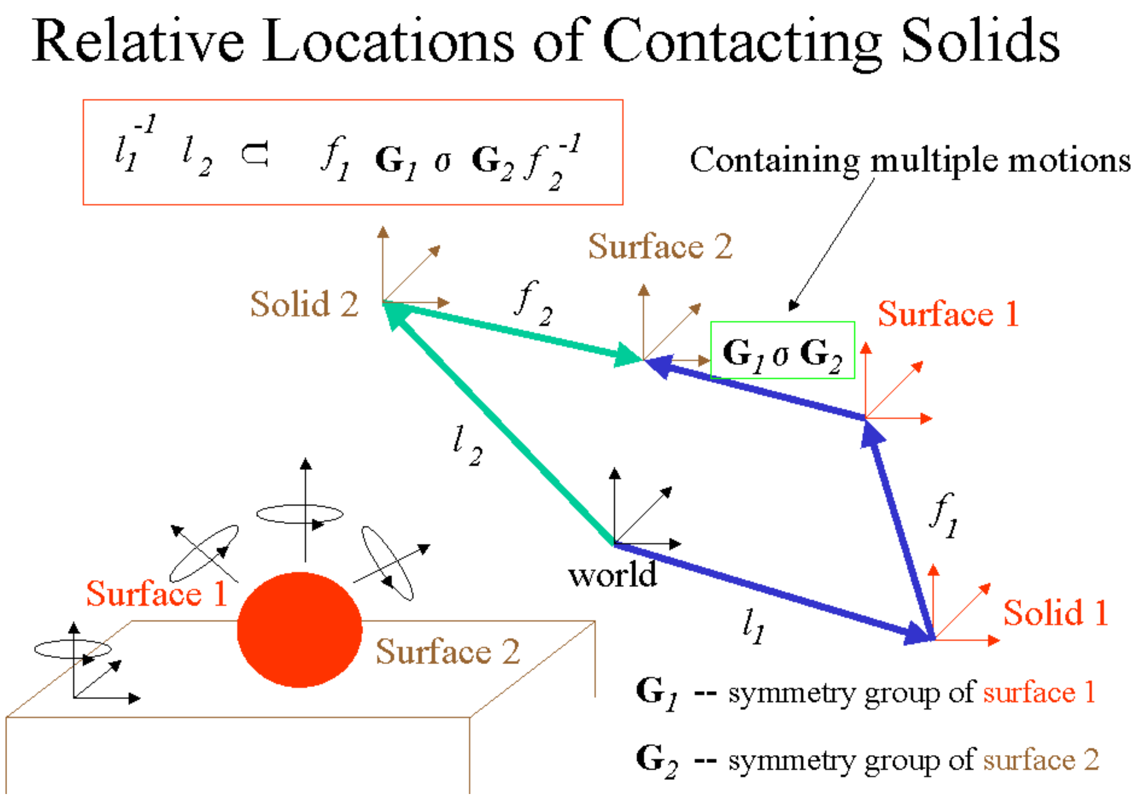

Figure 1: The relative locations of two solids (B_1, B_2) in contact through their surfaces F_1, F_2, are expressed in terms of their symmetry groups G_1, G_2: $l_1^{-1}l_2 \in f_1 G_1 \sigma G_2 f_2^{-1}$, where l_1, l_2 specify the locations of solids B_1, B_2 in the world coordinate system and f_1 and f_2 specify the locations of F_1, F_2 in their respective body coordinates. σ is a transformation bringing the two coordinates together.

there is a direct relationship between the relative locations of two solids in contact and the symmetry groups of their contacting surfaces. Furthermore, it can be proven [14] that the most basic group operation is symmetry group intersection. The computational *challenge* is to find out, on computers

1. How to denote symmetry groups, which can be finite, infinite, discrete or continuous sub-

groups of the proper Euclidean group $\mathcal{E}^+$? and

2. How to compute these subgroup intersections in Euclidean space (under different locations and orientations) efficiently (in non-exponential time)?

We have employed a geometric approach to denote and intersect an important family of subgroups of the Euclidean group $\mathcal{E}^+$. They are called **TR** groups, defined as a semidirect product $G = TR$, where T and R are translation and rotation subgroups of $\mathcal{E}^+$ respectively. By mapping a **TR** group to a pair of translation and rotation characteristic invariants, the intersection of two subgroups can be done geometrically. We have developed and implemented a group intersection algorithm which is proven to be correct and efficient (Figure 2).

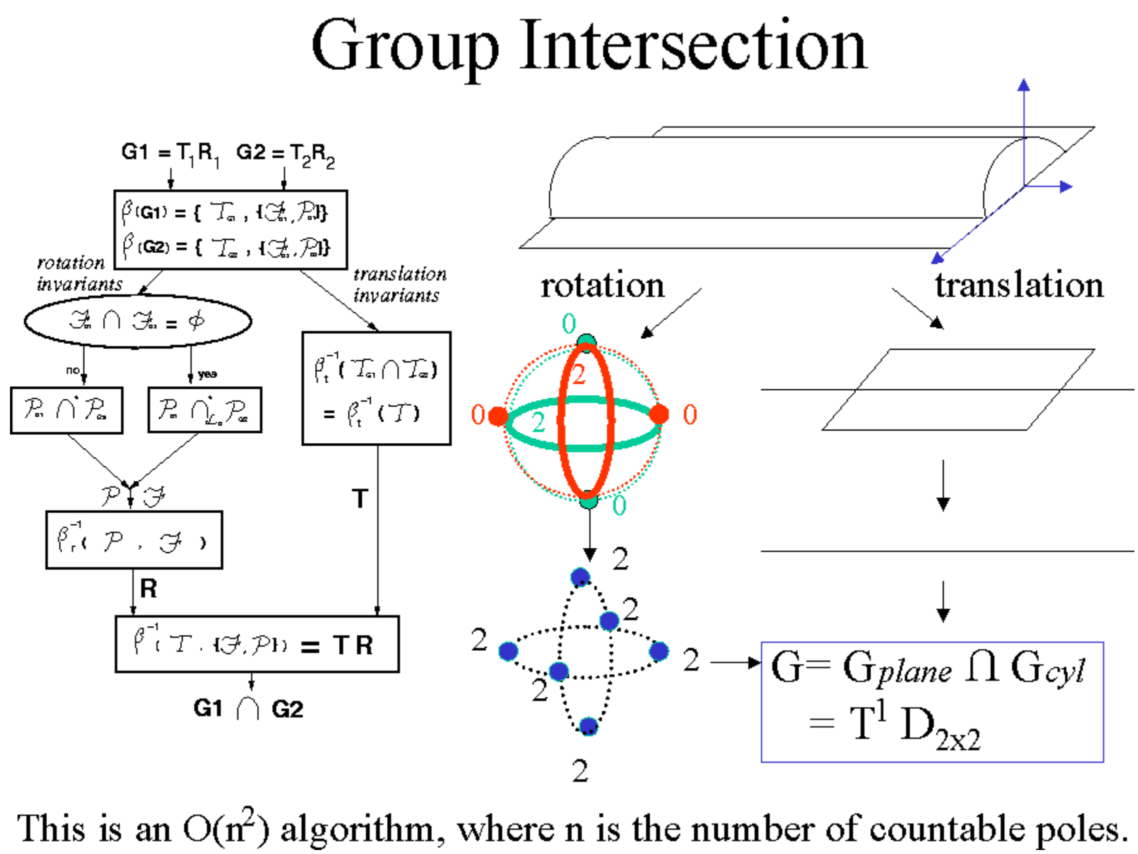

Figure 2: Left: **TR** group intersection algorithm. Right: An example of the intersection of two **TR** groups: symmetry groups of a plane and a cylinder

As an application platform for our group theoretical formalization of surface contact among solids, a Kinematic Assembly planning system $\mathcal{KA}3$ (Figures 3,4) [24, 25, 26, 27, 28] has been implemented. A designed assembly is input to $\mathcal{KA}3$ as a set of CAD models (boundary files) of individual parts and symbolic relationships among them. $\mathcal{KA}3$ generates a partial ordered precedence graph with symmetry groups and homogeneous transformation matrices attached to each contact. Note, the contact can be either fixed, when the symmetry group of the contacting surfaces is the identity, or having relative motions when the symmetry group is non-trivial.

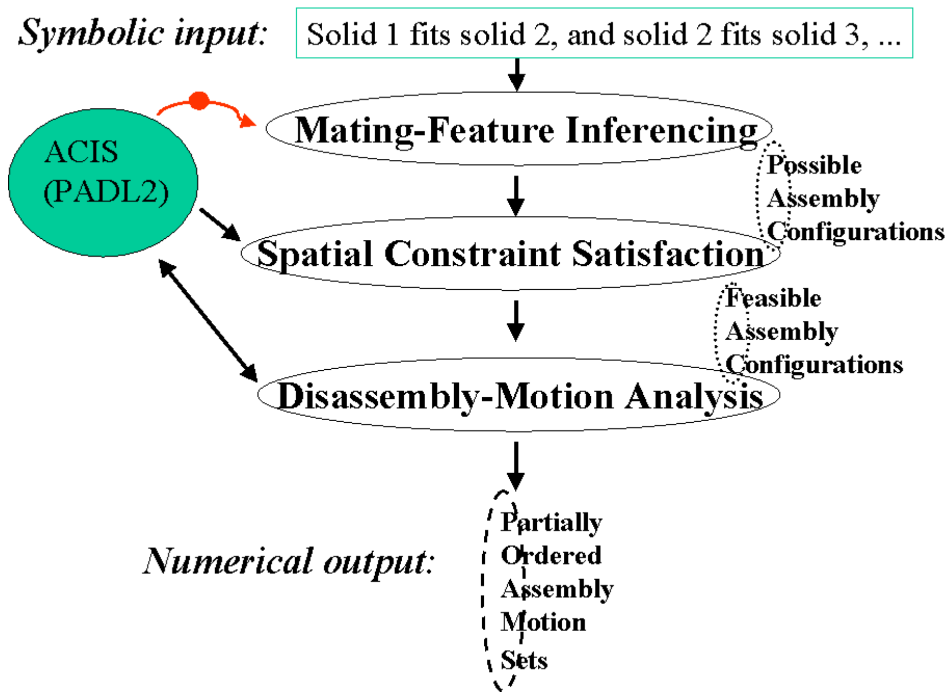

Figure 3: The structural framework of a Kinematic Assembly Planning system ($\mathcal{KA}3$) where symmetry groups are used for reasoning about solids in contacts. The output is an assembly plan for robotic execution.

2.2 Pathological Neuroradiology Image Indexing and Retrieval via Quantification of Brain (a)Symmetry

Normal human brains exhibit an approximate bilateral symmetry with respect to the interhemispheric (longitudinal) fissure bisecting the brain, known as the anatomical **midsagittal plane** (MSP). However, human brains are almost never perfectly symmetric [2, 3, 7]. Pathological brains, in particular, often depart drastically from perfect reflectional symmetry. For effective pathological brain image alignment and comparison in a large pathological medical image database (e.g., [2, 21, 22, 29]), it is most desirable to define a *plane of reference* that is invariant for symmetrical as well as asymmetrical brain images and to develop algorithms that capture this reference plane robustly.

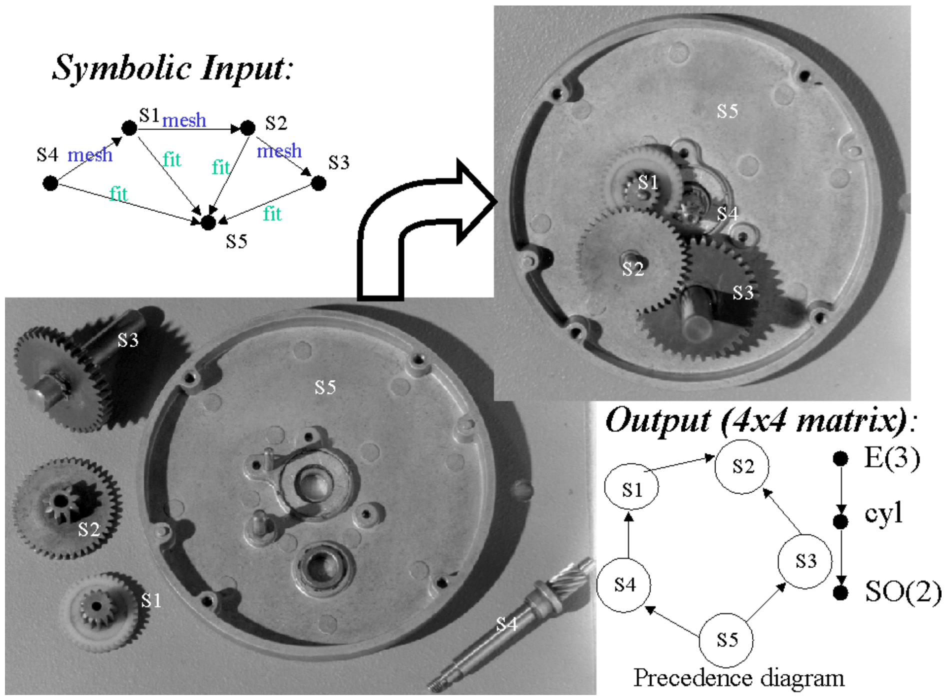

Figure 4: $\mathcal{KA}3$ figures out the spatial and kinematic relations of a gearbox.

We have developed an algorithm that is capable of finding an ideal MSP (iMSP) from a given volumetric pathological neuroimage [18, 20]. The goal here is to find where the iMSP is supposed to be if the brain had not been deformed due to internal brain asymmetry, pathology or external initial position/orientation offsets, noise[1], and bias fields. The tolerance of our iMSP extraction algorithm to these internal and external factors in various volumetric neuroimages is demonstrated in Figure 5.

After the iMSP is identified for each 3D brain image, we have achieved simultaneously an alignment of different 3D neuroimages and a baseline for extracting useful image features for comparing different brains and pathologies. Figure 6 shows how a set of quantitative measurements of brain asymmetry can be computed. By using 50 asymmetry measurements of each brain (Figure 7), we have constructed an image retrieval system to find most similar images in the database for a given query image. Figure 8 displays two sample retrieval results. The system achieves around 80% average true positive rate during retrieval [21, 22].

[1]The noise is measured by SNR or Signal to Noise Ratio, which is defined as 10 * log(var(signal)/var(noise)). An SNR of less than 0 means that the noise has a higher variance than the signal.

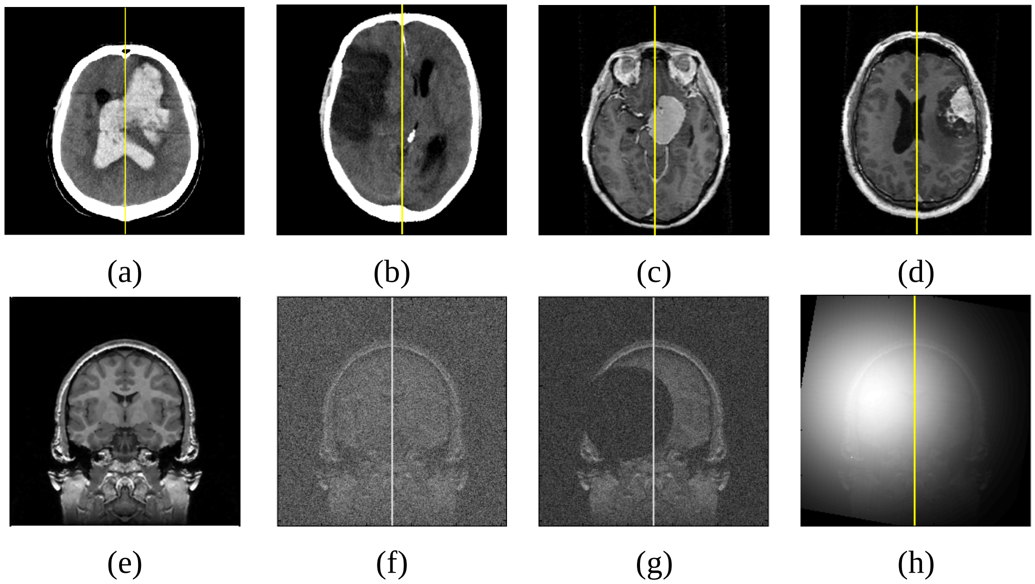

Figure 5: The ideal midsagittal planes (iMSP) extracted from different clinical 3D CT ((a),(b)) and MR images ((c),(d)). The 2D line is the intersection of the iMSP and the given 2D brain slice. (e) one MR brain slice without noise. (f) on a dataset with added noise. SNR of breaking point is -10.84dB. (g) on a dataset with an artificial lesion plus noise. SNR of breaking point is -4.82dB. (h) on a dataset with added bias field G = 10, from which our algorithm still finds the iMSP correctly.

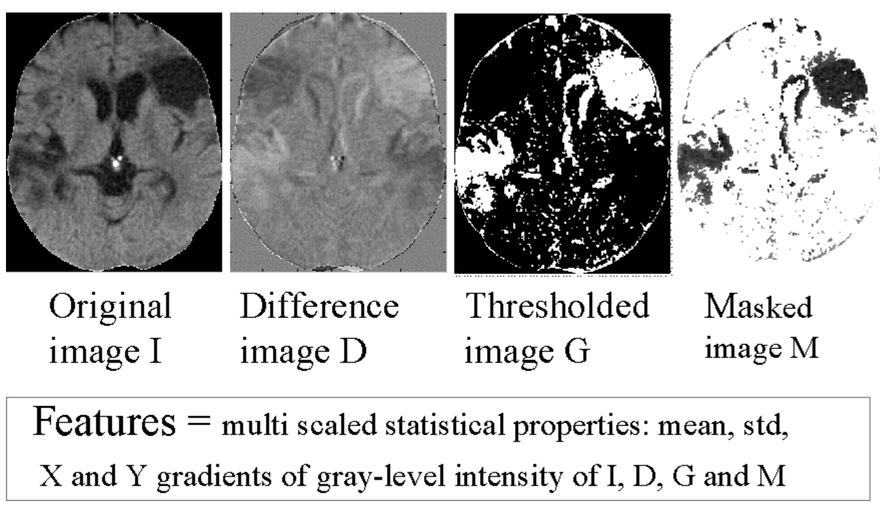

Figure 6: A set of statistical asymmetry measurements can be computed from neuroimages where the iMSP is found and aligned in the middle of the image.

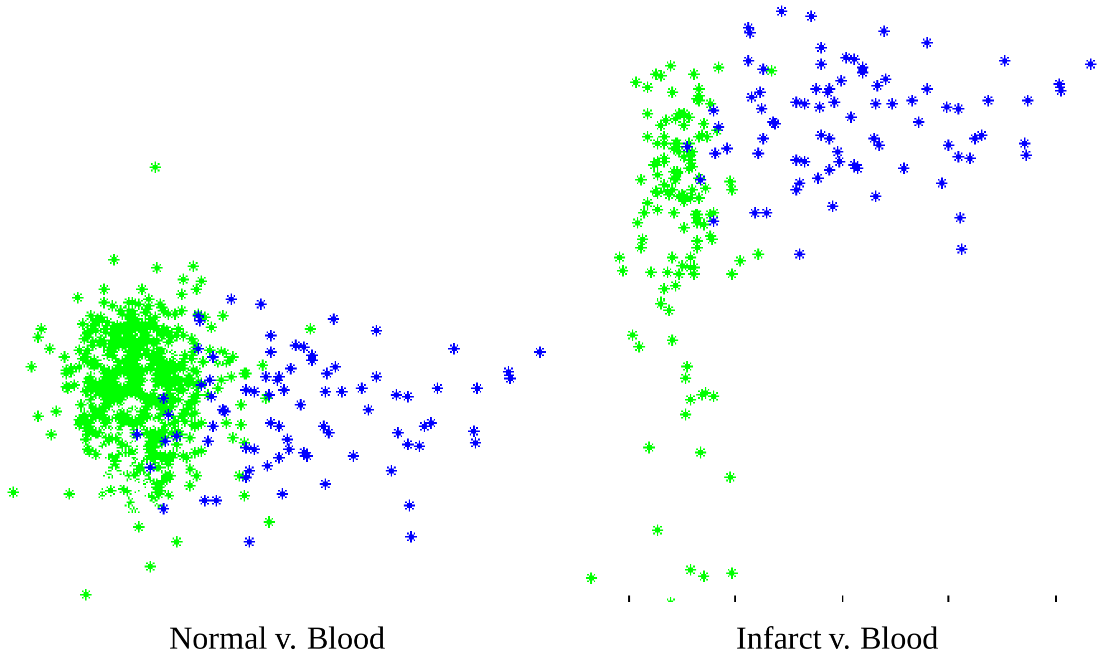

Figure 7: The 50 dimensional asymmetry measurements of each brain in the database are projected onto a plane. Separations can be observed between the distributions of asymmetry measurements of normal and blood, infarct and blood typed brains.

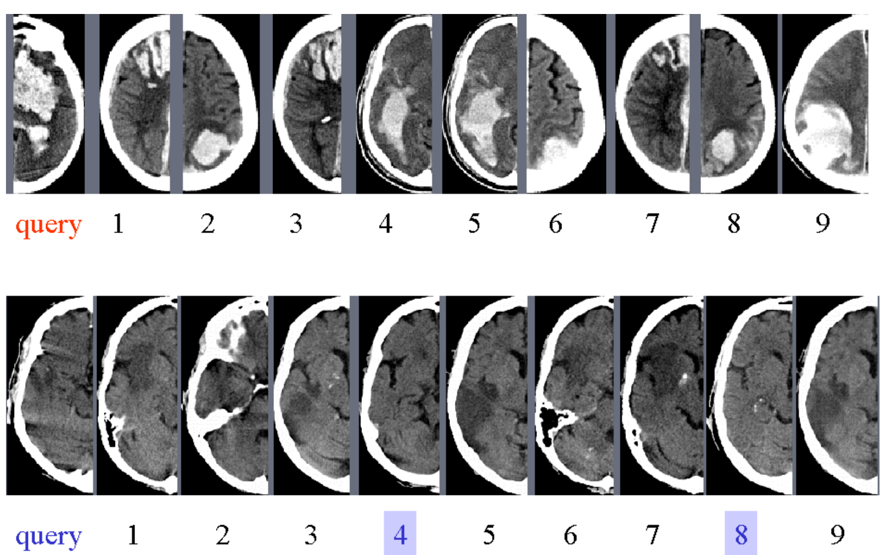

Figure 8: Classification-driven semantic-based image retrieval results: the top nine most similar images to the query image drawn from a database of 1200 images. The query image on the top is an acute blood case and the one at the bottom is an infarct case. Shaded labels indicate misclassified images.

2.3 A Computational Model for Periodic Pattern Perception based on Crystallographic Groups

A mature mathematical theory for periodic patterns has been known for over a century [1, 4, 8], namely, the crystallographic groups. These are groups composed of symmetries of periodic patterns in n dimensional Euclidean space. The amazing result is that regardless of the dimension n and the fact that there are infinite possible periodic patterns, the number of symmetry groups for periodic patterns in that space is always finite [1]! In particular, for monochrome planar periodic patterns, there are seven *frieze groups* [12, 33] for 2D patterns repeated along one dimension (strip patterns), seventeen *wallpaper groups* [30] describing patterns extended by two linearly independent translational generators (wallpaper patterns), and 230 space groups [11, 6] extended by three linearly independent translations (regular crystal patterns).

It is the goal of this research to construct a computational model for periodic pattern perception and analysis based on the theory of crystallographic groups. Given the digital form of a periodic pattern, a computer can discover its underlying lattice, its symmetry group, its motifs, **and** what other symmetry groups it can be associated with when the pattern undergoes affine deformations [16, 19].

2.3.1 Automatic Lattice Construction

Autocorrelation of a given periodic pattern, which may only contain 2-3 cycles and lots of noise, is used to detect the underlying lattice structure. Even noise-free computer-generated patterns can cause problems for lattice detection algorithms. Halfway between actual lattice translations, the large sub-patterns may partially match smaller sub-patterns interspersed between them, causing spurious peaks to form. Furthermore, these spurious peaks can have higher value than actual peaks located at the periphery of the autocorrelation image. Figure 9(A) shows an autocorrelation surface for the rug image on the left. Although the grid of peaks is apparent to the human eye, finding it automatically is very difficult. Simple approaches such as setting a global threshold yield spurious results (Figure 9(B)). We used a novel peak detection algorithm based on "regions of dominance" [16] to automatically detect the underlying translational lattice, its result on the rug is shown in Figure 9(D).

The trouble is that many legitimate grid peaks have a lower value than some of the spurious peaks. Figure 9(C) presents the first 32 peaks found by our peak detection algorithm. 9(D) shows the formed lattice.

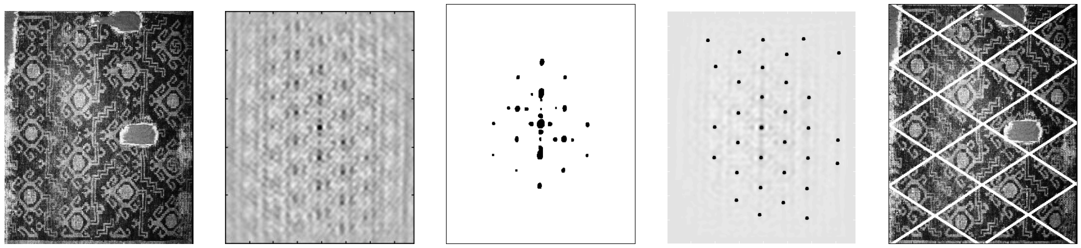

Figure 9: An oriental rug image and (A) its autocorrelation surface, (B) peaks found using a global threshold, (C) the 32 most-dominant peaks found using our approach described in the text. (D) detected lattice

2.3.2 Symmetry Group Classification

Table 1 lists the eight symmetries checked in the classification algorithm. The determination of a specific rotation or reflection or glide-reflection symmetry is performed with respect to the unit lattice orientations found, by applying the symmetry to be tested to the entire pattern, then checking the similarity between the original and transformed images.

It is interesting to notice the difference between the symmetry group classification flowcharts for humans [30, 34] and the computer's classification algorithm: the computer has to first find the underlying lattice structure (not necessarily where it is anchored however) while humans do that implicitly and the first question for humans is: *what is the smallest rotation?*

2.3.3 Automatic Motifs Generation

Choosing a good motif should help one see, from a single tile, what the pattern "looks like" (Figure 10). From work in perceptual grouping, it is known that the human perceptual system often has a preference for symmetric figures [13]. If we entertain the idea that the most representative motif is the one that is most symmetrical, one plausible strategy for generating motifs is to align the motif center with the center of the highest-order rotation in the pattern. Candidate motifs can then be determined systematically by enumerating each distinct center point of the highest-order rotation. Two rotation centers are distinct if they lie in different **orbits** of the symmetry group [16]. Figure 11 shows a set of symmetrical motifs from periodic patterns of various symmetry groups. *Approximate symmetries* in a pattern are used to fix the unconstrained lattice structure for symmetry groups like pm, pg and cm or $p1$ that do not have rotation centers [16]. Aside from motif selection, knowledge of the lattice structure of a repeated pattern allows us to determine which pixels in an image should look the same. Taking the median of corresponding pixels across

Table 1: Wallpaper group classification: numbers 2,3,4 or 6 denote n-fold rotational symmetry, Tx (or Dx) denotes reflectional symmetry about one of the translation (or diagonal) vectors of the unit lattice. “Y” means the symmetry exists for that symmetry group; empty space means no. Y(g) denotes the existence of a glide reflection.

	p1	p2	pm	pg	cm	pmm	pmg	pgg	cmm	p4	p4m	p4g	p3	p3m1	p31m	p6	p6m
2		Y				Y	Y	Y	Y	Y	Y	Y				Y	Y
3													Y	Y	Y	Y	Y
4										Y	Y	Y					
6																Y	Y
T1			Y	Y(g)		Y	Y(g)	Y(g)			Y	Y(g)			Y		Y
T2						Y	Y	Y(g)			Y	Y(g)			Y		Y
D1					Y				Y		Y	Y		Y	Y		Y
D2									Y		Y	Y					Y

the multiple tiles of the rug image, for example, creates a “median tile” with noise and irregularities filtered out. Figure 12 compares the original worn rug with a virtual rug generated from the median tile.

2.3.4 Skewed Symmetry Groups

Table 2 is a transition matrix where each entry indicates whether the row-group can be affinely transformed into a column-group. It is discovered that symmetry groups of periodic patterns form small orbits (2-4 members) when they are affinely skewed (Figure 13). A classification algorithm is developed to evaluate the potential skewed symmetry groups of a given pattern [16, 19, 17]. The practical value of this result in computer vision includes a new principled measure for potential symmetry, indexing and retrieval of regular patterns, and estimation of shape and orientation from texture[2].

3 Conclusion

We have dealt with three very different types of applications using the concept of symmetry and group theory. Figure 14 provides a comparative view of their relations in terms of the data, method and the complexity of symmetry treated.

[2] Some new result has been submitted to the *International Conference of Computer Vision in 2001*

Table 2: **Wallpaper Group Transition Matrix Under Affine Transformations** **S** – similarity transformation, **N** – non-uniform scaling ⊥ or || to all reflection axes in the group to the left, **A** – general affine transformation other than **S** or **N**, **P** – depending on the particular pattern, and numbers – a labeled justification on why no affine deformation that can possibly transform the row-group to the column-group [17]

	p1	p2	pm	pg	cm	pmm	pmg	pgg	cmm	p4	p4m	p4g	p3	p3m1	p31m	p6	p6m
p1	**A**	1	**P**	**P**	**P**	1	1	1	1	1	1	1	**P**	**P**	**P**	1	1
p2	1	**A**	1	1	1	**P**	**P**	**P**	**P**	**P**	**P**	**P**	1	1	1	**P**	**P**
pm	**A**	1	**N**	2	3	1	1	1	1	1	1	1	4	3	3	1	1
pg	**A**	1	2	**N**	3	1	1	1	1	1	1	1	4	3	3	1	1
cm	**A**	1	3	3	**N**	1	1	1	1	1	1	1	4	**P**	**P**	1	1
pmm	1	**A**	1	1	1	**N**	2	2	**P**	3	**P**	3	1	1	1	4	3
pmg	1	**A**	1	1	1	2	**N**	2	3	3	3	3	1	1	1	4	3
pgg	1	**A**	1	1	1	2	2	**N**	**P**	3	3	**P**	1	1	1	4	3
cmm	1	**A**	1	1	1	**P**	3	**P**	**N**	3	**P**	**P**	1	1	1	3	**P**
p4	1	**A**	1	1	1	3	3	3	3	**S**	2	2	1	1	1	4	3
p4m	1	**A**	1	1	1	**N**	3	3	**N**	2	**S**	2	1	1	1	3	3
p4g	1	**A**	1	1	1	3	3	**N**	**N**	2	2	**S**	1	1	1	3	4
p3	**A**	1	4	4	4	1	1	1	1	1	1	1	**S**	2	2	1	1
p3ml	**A**	1	3	3	**N**	1	1	1	1	1	1	1	2	**S**	2	1	1
p3lm	**A**	1	3	3	**N**	1	1	1	1	1	1	1	2	2	**S**	1	1
p6	1	**A**	1	1	1	4	4	4	3	4	3	3	1	1	1	**S**	2
p6m	1	**A**	1	1	1	3	3	3	**N**	3	3	4	1	1	1	2	**S**

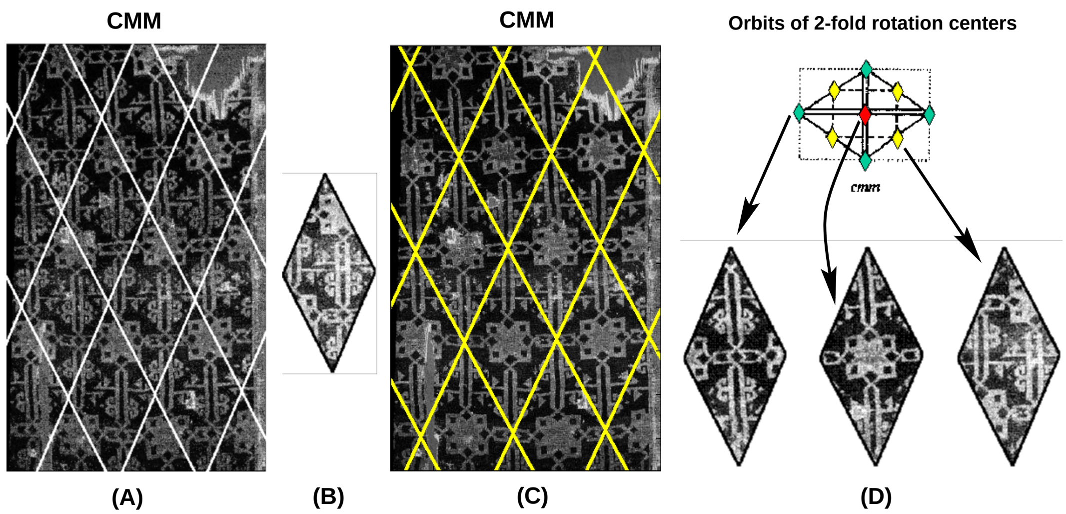

Figure 10: (A) and (B) show an automatically extracted lattice and the tile that it implies. The tile is not a good representation of the pattern motif. (C) and (D) show the lattice positioned in one of the three orbits of 2-fold rotation centers in symmetry group cmm, (D) displays the three most-symmetric motifs found.

Our research seems to suggest that symmetry as a computational concept is not simply a binary valued variable, nor only varies in a single dimension. Figure 15 demonstrates the author's view of symmetry in practice: **symmetry spans a multi-dimensional, continuous space**.

We have presented a sampler of the author's research activities towards automation of symmetry analysis using computers. These results show that symmetry and group theory can play an important role in solving practical problems. The concept of symmetry often offers the key insight that resolves seemingly tedious, messy and even random factors in a problem (e.g. assembly parts relationships, pathological brains indexing, real world periodic pattern analysis). The main challenge in *computational symmetry* is how to construct plausible computational tools to automate the transition from abstract symmetry concepts and group theory to realistic applications. It has been and will continue to be a challenging yet rewarding process.

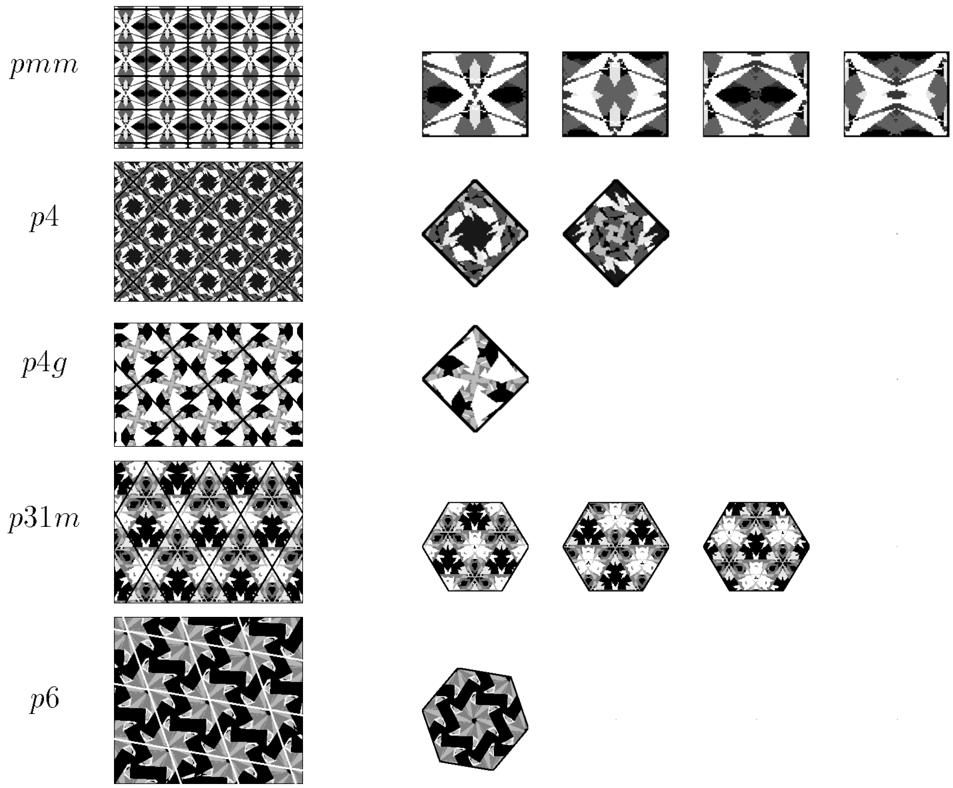

Figure 11: Automatically detected lattices and motifs for some of the seventeen wallpaper groups.

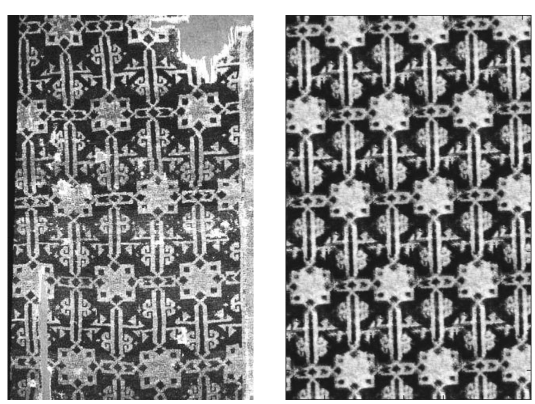

Figure 12: Real oriental rug and a perfectly symmetric virtual rug formed by translating the median tile.

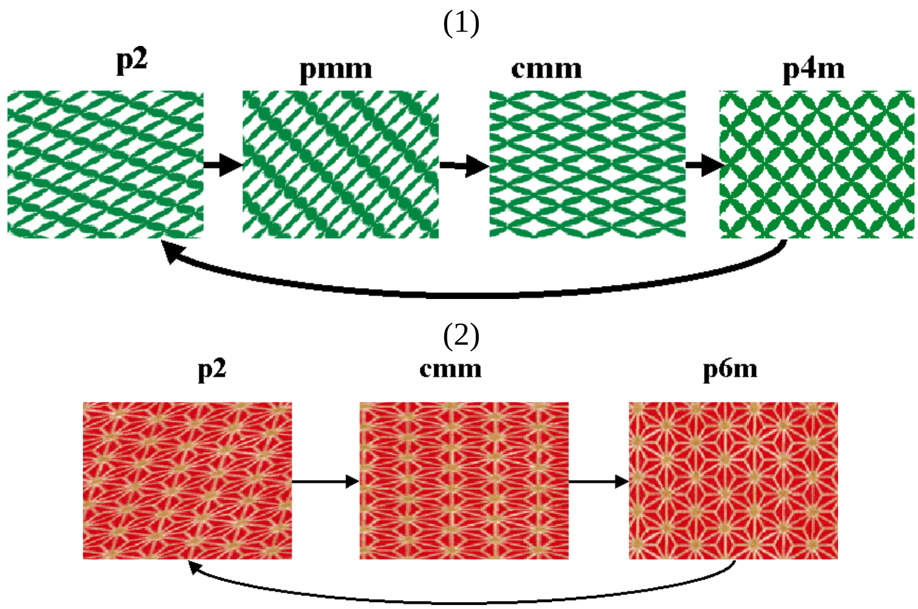

Figure 13: While the pattern is deformed by affine transformations its symmetry group migrates to different groups within its orbit: (1) $p2 \to pmm \to cmm \to p4m$, (2) $p2 \to cmm \to p6m$,

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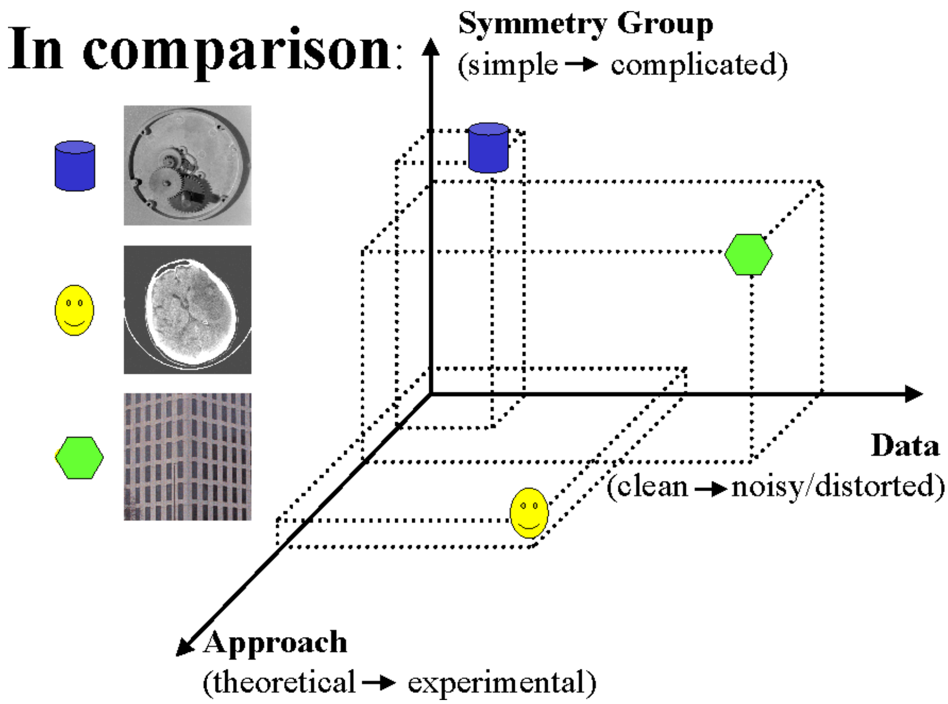

Figure 14: A perspective view of the three examples reported

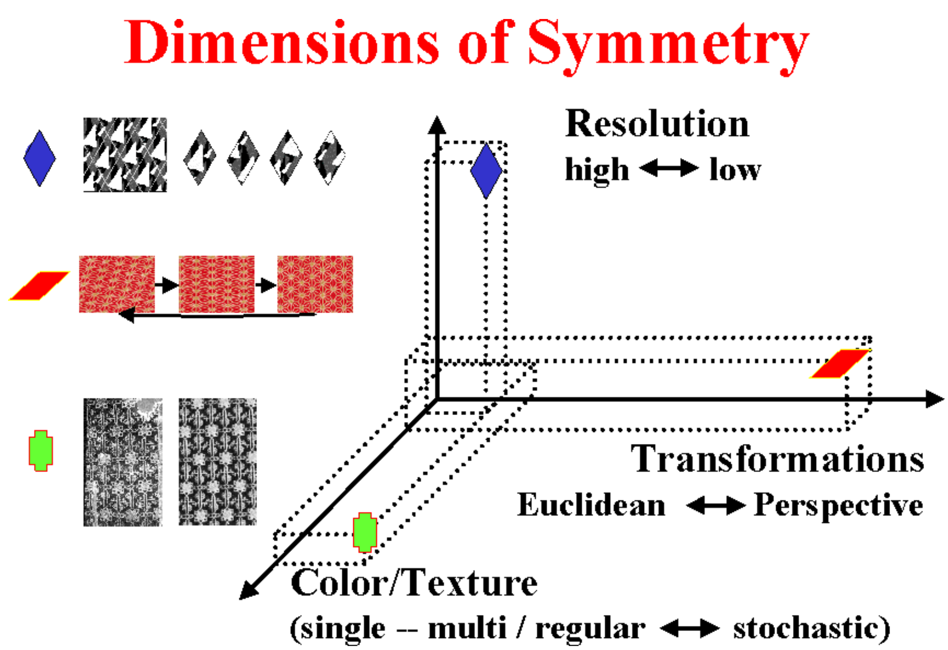

Figure 15: Symmetry spans a continuous, multi-dimensional space

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