Connected Invariant Sets for High-Speed Motion Planning in Partially-Known Environments

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***Abstract*— Ensuring safety in partially-known environments is a critical problem in robotics since the environment is perceived through sensors and the environment cannot be completely known ahead of time. Prior work has considered the problem of finding positive control invariant sets (PCIS). However, this approach limits the planning horizon of the motion planner since the PCIS must lie completely in the limited known part of the environment. Here we consider the problem of guaranteeing safety by ensuring the existence of at least one PCIS in partially-known environments leading to an extension of the PCIS concept. It is shown, that this novel method is less conservative than the common PCIS approach and robust to unknown small obstacles which might appear in the close vicinity of the robot.**

An example implementation for loiter circles and power line obstacles is presented. Simulation scenarios are used for validating the proposed concept.

I. INTRODUCTION

Many robotic challenges such as navigation of autonomous cars or unmanned aerial vehicles (UAVs) require to perceive the world with sensors to get the information about their environment. However, each perception system has limited capabilities due to limited sensor range, occlusions by obstacles or sensor noise which provides the robot only with incomplete information of its surrounding resulting in a so called partially-known environment (PKE). Robots operating in PKEs must take into account that not all obstacles in the surrounding of the robot are detected at the same distance. The detection range depends on various properties of the object or the used perception system. But in general the following rule applies: the smaller the obstacle, the shorter the detection distance. Thus, small obstacles might be detected only in a short distance to the robot constituting a safety issue. An illustrative example is an UAV that is equipped with a LIDAR and is flying in partially-known environments (Fig. 1), which might contain power lines. Due to the small profile of the power lines, they are only detectable in the vicinity of the UAV. Since the obstacles are small, they only require small changes in the robot motion in order to prevent collisions. We want to formalize this notion and find alternatives that enable guarantees on the safety of the vehicle with unknown obstacles.

Common approaches for navigation in PKEs use model predictive control (MPC) [1] also called receding horizon control (RHC). The main idea of RHC is to apply iterative optimization inside a finite time horizon of the known space. Thereby, the problem is partitioned into subproblems which allow an on-line optimization. However, two problems arise from the partial trajectories: the problem of stability (in the sense of reaching a goal) and the problem of feasibility (no constraint violation). The common technique to ensure feasibility is to apply the theory of positive control invariant sets (PCIS) [2]. If the system is inside a PCIS it can stay there for an infinite time ensuring safety. However, the theory of PCIS requires complete knowledge about its environment. In order to apply PCIS to PKEs, it must be ensured that the PCIS lies completely inside the known space of the environment. Thus, the PCIS is limiting the planning horizon of the motion planner due to its strict constraints, especially for robotic systems operating at high speeds such as UAVs.

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Fig. 1: Automatic landing of the Unmanned Little Bird (ULB).

In this work, the problem of feasibility in PKEs with the application to fast flying UAVs is discussed. Therefore, an extension of PCIS is presented which is robust to unknown obstacles in the environment. An on-line capable implementation is presented which uses a set of connected PCISs to ensure the existence of at least one PCIS regarding a finite set of unknown obstacles. Finally, simulation results are presented verifying that this novel approach relaxes the constraints for the motion planner and hence improves the overall navigation result of the UAV by allowing higher velocities.

II. RELATED WORK

In order to guarantee feasibility, no violation of constraints including the collision constraint, *invariant set theory* [3] is used in RHC applications. In [4] a *non-linear model predictive control* framework is presented which guarantees nominal feasibility using invariant sets. In order to guarantee feasibility at all times, necessary and sufficient conditions on the control horizon, prediction horizon and constraint state space are presented. In [5] receding horizon control for UAVs based on mixed integer linear programming is presented. Therefore, loiter circles represented by affine transformations are added to the mixed integer linear programming formulation to ensure safety for an infinite time horizon. The main difference to former approaches is, that this method is applied on-line and needs no complex off-line computation which is not flexible to changes in the environment. A library of precomputed emergency loiter maneuvers is used in [6] to ensure safety for a full-scale autonomous helicopter. It generates a set of feasible emergency maneuvers for the discretized state space of the helicopter by optimizing the path diversity, i.e. the difference of paths according to some measure. As a result, this approach can on-line assess the safety of the helicopter at flights over $50\,\mathrm{m/s}$.

In order to ensure feasibility of the robotic system under bounded disturbances, robust positive control invariant sets (RPCIS) are used. Therefore, constraint tightening is used in [7] to take into account disturbances such as changing wind. In [8] a robust MPC using MILP formulation is presented which is an extension of the work in [5]. The two approaches, constraint tightening and affine feedback reparameterization, are evaluated to take into account disturbances. The idea of tube-based MPC [9] also relies on RPCIS to solve the robust MPC problem.

Other related work deals with the problem of the existence of an infinitely long collision-free trajectory. This problem is discussed in [10] by the means of the ergodic forest scenario. The aim is to determine criteria such as size of the obstacles, distribution of the obstacles and maximum velocity of the robot under which almost surely an infinite time trajectory exists.

The problem of feasibility is also addressed by the concepts of *region of inevitable collision* (RIC) [11] and *inevitable collision states* (ICS) [12]. The RIC is defined as the set of states which are already in collision or which are inevitably leading to a collision. In [13] the analytic computation of RIC for convex polygons and the approximation for unions of RIC sets is discussed with the aim to make motion planning more efficient and safer. The effectiveness of the RIC approach for motion planning of vehicles with underactuated dynamics is shown in [14]. The concept of ICS has already been applied in dynamic environments [15] and also for car-like vehicles [16, 17]. Additionally, in [18, 19] an extension for stochastic environments is presented taking into account the uncertain motion prediction of the obstacles in the workspace. Therefore, the probability is calculated that a state of the robot will inevitably lead to a collision.

However, to the best knowledge of the authors, no related work exists discussing the robustness of PCISs regarding unknown obstacles in PKEs. In the next section, the problem formulation for ensuring feasibility in partially-known environments is given.

III. PROBLEM FORMULATION

In this section we formulate the problem of guaranteeing safety for an UAV in partially-known environments resulting from the limited perception capabilities. Furthermore, we only consider static environments and assume that the robot has deterministic behavior meaning that there exists no motion uncertainty.

A. Robot Model and Environment Description

The state of the robot at a certain time point t is denoted as $\mathbf{x}(t) = [\mathbf{p}(t), \boldsymbol{v}(t), \theta(t)]$ and contains the position $\mathbf{p}(t) = [x(t)\ y(t)\ z(t)]$ and velocity $\boldsymbol{v}(t) = [\dot{x}(t)\ \dot{y}(t)\ \dot{z}(t)]$ and roll $\theta(t)$ given in an inertial coordinate frame. The dynamics of the robot are characterized by the motion model $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t))$ and the constraint input space $u(t) \in \mathcal{U}$ and state space $\mathbf{x}(t) \in \mathcal{X}$. It is noted, that the proposed definition and concepts of the novel safety assessment approach are independent of the used motion model. The workspace of the robot is denoted by $\mathcal{W}$, $\mathbf{p}(t) \in \mathcal{W}$ and the subset of the workspace occupied by the robot is referred to $\mathcal{A} \subset \mathcal{W}$ and as $\mathcal{A}(\mathbf{x}(t))$ at the state $\mathbf{x}(t)$. The occupancy of the ith obstacle in the workspace is denoted by $\mathcal{B}_i$, whereas all obstacles are considered as static.

Since we consider a partially-known environment we distinguish between known $\mathcal{B}^{\mathrm{k}}$ and unknown obstacles $\mathcal{B}^{\mathrm{u}}$. The unified occupancy of all objects is written in short notation as $\mathcal{B} = \mathcal{B}^{\mathrm{k}} \cup \mathcal{B}^{\mathrm{u}}$. The limitation of the perception system is modeled by two different field of view (FOV) regions $\mathrm{FOV}^{\mathrm{k}} \subset \mathrm{FOV}^{\mathrm{u}} \subset \mathcal{W}$. This models the fact, that large obstacles can be detected at a longer distance ($\mathrm{FOV}^{\mathrm{u}}$) than for smaller obstacles. Any obstacle which is inside $\mathrm{FOV}^{\mathrm{k}}(\mathbf{x}(t))$ at state $\mathbf{x}(t)$ is assumed to be detected by the perception system. In $\mathrm{FOV}^{\mathrm{u}}(\mathbf{x}(t))$ only obstacles with cross section $\varnothing$ bigger than $\tau_{\varnothing}$ will be detected. Thus, we distinguish between two workspace sets

$$\mathcal{W}^{\mathrm{k}} = \bigcup_t \mathrm{FOV}^{\mathrm{k}} \subset \mathcal{W}, \quad \mathcal{W}^{\mathrm{u}} = \bigcup_t \mathrm{FOV}^{\mathrm{u}} \subset \mathcal{W},$$

where $\mathcal{W}^{\mathrm{k}}$ is the part of the workspace in which all obstacles are known and $\mathcal{W}^{\mathrm{u}}$ represents the partially-known workspace. Depending on the distance from the robot to the obstacle and its cross section $\varnothing$ the obstacle is known or unknown

$$\mathcal{B}_i \in \begin{cases} \mathcal{B}^{\mathrm{k}}, \exists t\ \mathcal{B}_i \cap \mathcal{W}^{\mathrm{k}}(t) \neq \emptyset, \\ \mathcal{B}^{\mathrm{k}}, \exists t\ \mathcal{B}_i \cap \mathcal{W}^{\mathrm{u}}(t) \neq \emptyset \wedge \varnothing(\mathcal{B}_i) \geq \tau_{\varnothing}, \\ \mathcal{B}^{\mathrm{u}}, \mathrm{otherwise}, \end{cases}$$

where $\tau_{\varnothing}$ is the threshold of the cross section and $\mathcal{W}^{\mathrm{k}}(t), \mathcal{W}^{\mathrm{u}}(t)$ are the known and partially-known subsets of the workspace at time t. The obstacles $\varnothing(\mathcal{B}_i) < \tau_{\varnothing}$ represent obstacles which have a cross section below the threshold and thus are hard to detect for the perception system. Regarding the application of UAVs, typical examples for such kind of obstacles are wires, conductors or thin utility poles. Only in the near vicinity ($\mathrm{FOV}^{\mathrm{k}}$) of the robot the detection is guaranteed. Furthermore, a finite set of $\mathcal{B}^{\mathrm{u}}$ inside $\mathcal{W}^{\mathrm{u}}$ is assumed, meaning that only a maximum number of obstacles may appear inside $\mathcal{W}^{\mathrm{u}}$.

B. Problem Statement

In deterministic (completely known) environments, safety beyond the planning horizon is guaranteed by ensuring that the considered state of the robot is inside a positive control invariant set:

Definition 1 (Positive Control Invariant Set (PCIS) [2]): *The set $\Omega \subset \mathcal{X}$ is a positive control invariant set if it is obstacle-free and dynamically feasible meaning $\forall \mathbf{x}(t=0) = \mathbf{x}_0 \in \Omega$ there exists a continuous feedback control law $u(t) = \Phi(\mathbf{x}(t)), u \in \mathcal{U}$ which assures $\mathbf{x}(t) \in \Omega \wedge \mathcal{A}(\mathbf{x}(t)) \cap \mathcal{B}^{\mathrm{k}} = \emptyset$ for $t > 0$.*

The PCIS guarantees that the robot can stay inside Ω for an infinite time without collisions. Therefore, the PCIS must be completely inside of the known workspace $\Omega \subset \mathcal{W}^{\mathrm{k}}$. Examples of such feasible invariant sets are braking trajectories or loiter patterns. In this work the problem of invariant sets in *partially-known environments* is discussed. First, we give the definition of the maximal positive control invariant set from literature:

Definition 2 (Maximal Positive Control Invariant Set (MPCIS) [4]): *The non-empty set $C^{\infty}(\mathcal{X})$ is the maximal control invariant set contained in $\mathcal{X}$ for the system $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t))$ if and only if $C^{\infty}(\mathcal{X})$ is positive control invariant and contains all positive control invariant sets contained in $\mathcal{X}$, i.e. Ω is positive control invariant only if $\Omega \subseteq C^{\infty}(\mathcal{X}) \subseteq \mathcal{X}$. This set can be seen as a set of PCISs $\mathcal{C}^{\infty}(\mathcal{X}) = \{\Omega_1, \Omega_2, \dots, \Omega_{\mathrm{N}_\Omega}\}$.*

This definition allows us to formulate the feasibility problem in PKEs

Problem 1 (Feasibility in PKEs): *Feasibility in PKEs is ensured iff*

$$\nexists \mathcal{B}^{\mathrm{u}}, \ |\mathcal{B}^{\mathrm{u}}| = \mathrm{N}_{\mathcal{B}^{\mathrm{u}}} : \mathcal{C}^{\infty}(\mathcal{X}^{\mathrm{a}}(\underbrace{\mathcal{B}^{\mathrm{k}} \cup \mathcal{B}^{\mathrm{u}}}_{\mathcal{B}})) = \emptyset$$

with

$$\mathcal{X}^{\mathrm{a}}(\mathcal{B}) = \{\mathbf{x} \mid \mathbf{x} \in \mathcal{X} \ \ \forall \mathcal{B}_i \in \mathcal{B} \ \ \mathcal{A}(\mathbf{x}) \cap \mathcal{B}_i = \emptyset\}.$$

Loosely speaking, if there exists no possible configuration of the finite set $\mathcal{B}^{\mathrm{u}}$ that the MPCIS is an empty set, feasibility of the robotic system is guaranteed, since at least one PCIS exists.

IV. CONNECTED POSITIVE CONTROL INVARIANT SETS

In this section a novel approach for ensuring feasibility in partially-known environments based on Prob. 1 is presented. The main idea is to approximate the MPCIS with a set of connected PCISs ensuring feasibility for any configuration of the finite set $\mathcal{B}^{\mathrm{u}} \in \mathcal{W}^{\mathrm{u}}$. The set $\mathcal{B}^{\mathrm{u}}$ contains obstacles which can only be detected in the close vicinity of the robot ($\mathrm{FOV}^{\mathrm{k}}$). It is assumed that the parametric model of their geometry is known, but their configuration (position and orientation) is unknown. Furthermore, we restrict their maximum number to ensure the existence of at least one PCIS while taking into account all possible configurations.

A. General Idea

In order to ensure feasibility for a state of the robot or the final state of an intended trajectory, we approximate the MPCIS by a set of connected PCISs. All PCISs are

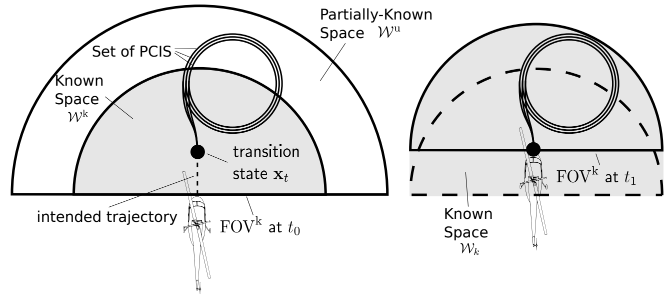

Fig. 2: The figure demonstrate the novel approach from the top view and shows the interaction between loiter circles, transition trajectories and known workspace. The FOV is presented by half circles and the loiter circles and the associated transition trajectories are illustrated by black lines. The transition state is depicted by a small solid black circle.

reachable from the common transition state $\mathbf{x}_t$. The individual PCIS are allowed to be partially outside the known space $\mathcal{W}^{\mathrm{k}}$ at the time of the assessment t_0. However, when the robot reaches the transition state, all PCISs must lie completely in the known area of the workspace at time t_1. Thus, the robot is able to choose a collision-free PCIS at the transition state. This concept of ensuring safety for an intended trajectory beyond the planning horizon is illustrated in Fig. 2.

B. Connected Set of PCISs

The computation of the MPCIS in the limited space $\mathcal{W}^{\mathrm{u}}$, would contain all possible PCISs, but its computation is only efficiently possible for LTI systems subject to linear inequality constraints, for piecewise affine systems and some classes of hybrid systems [4]. Therefore, we perform a conservative approximation by generating only a finite set of connected PCISs $\{\Omega_0, \dots, \Omega_{\mathrm{N}_\Omega}\}$. We assume that a common approach for the generation of a single PCIS Ω is used, e.g. [6]. All Ω need to respect the vehicle constraints and must be collision-free regarding the known obstacles in the workspace $\mathcal{W}^{\mathrm{u}}$. Furthermore, all Ω must contain a common state, the transition state $\mathbf{x}_t$ and all Ω must be inside the known space when the robot reaches $\mathbf{x}_t$

$$\begin{aligned} &\forall \Omega, \ \forall \mathbf{x} \in \Omega : \mathcal{A}(\mathbf{x}) \cap \mathcal{B}^{\mathrm{k}} = \emptyset \\ &\exists \mathbf{x}_t, \ \forall \Omega : \mathbf{x}_t \in \Omega \\ &\forall \Omega : \mathcal{W}^{\mathrm{u}} \supset \Omega \wedge \mathcal{W}^{\mathrm{k}}(\mathbf{x}_t) \supset \Omega, \end{aligned}$$

where $\mathcal{W}^{\mathrm{k}}(\mathbf{x}_t)$ is the known workspace when the robot reaches the transition state $\mathbf{x}_t$.

C. Robustness of Connected PCISs

In order to ensure feasibility of the set of PCISs $\{\Omega_1, \dots, \Omega_{\mathrm{N}_\Omega}\}$, one has to show that there exists no configuration of $\mathcal{B}^{\mathrm{u}}$ which can cover all Ω. This problem is related to the cycle elimination problem in graph theory, since the connected PCISs form a directed graph. Feedback arc sets or feedback vertex sets are used to solve for the cycle elimination problem [20]. The problem can also be transformed to a geometric set cover problem (SCP) [21]. The PCISs form the universe and each obstacle in $\mathcal{B}^{\mathrm{u}}$ can cover a subset of PCISs meaning that the execution of the PCIS will lead to a collision. If there exists a combination of subsets which covers the entire universe, there exists a solution to the set cover problem. The geometric set cover problem is formally defined as follows:

Problem 2 (Geometric Set Cover): *Given a finite set of* PCIS*s* $\{\Omega_1, \ldots \Omega_{\mathrm{N}_\Omega}\}$ *(called the universe) and a set*

$$\mathcal{F} = \{\mathcal{S}_1, \ldots, \mathcal{S}_{\mathrm{N_s}}\},\ \mathcal{S}_i \subset \{\Omega_1, \ldots \Omega_{\mathrm{N}_\Omega}\}$$

of subsets which can be covered by the considered geometric shapes, the set cover problem is to identify the set with the smallest cardinality $\mathcal{J}$ *which contains the entire universe (blocks all* Ω*)*

$$\min |\mathcal{J}| \text{ with } \mathcal{J} \subset \{1, \ldots, \mathrm{N_s}\}$$
$$\text{subject to } \forall \Omega,\ \exists \mathbf{x} \in \Omega,\ \exists i \in \mathcal{J} : \mathcal{A}(\mathbf{x}) \cap \mathcal{S}_i \neq \emptyset.$$

The set cover problem allows to formulate the proposition of the feasibility of connected PCISs:

Proposition 1 (Feasibility of Connected PCISs)**.** *A set of connected feasible loiter trajectories ensure feasibility iff the solution to the respective geometric set cover problem requires a cardinality greater than the cardinality of* $\mathcal{B}^{\mathrm{u}}$

$$|\mathcal{J}| > |\mathcal{B}^{\mathrm{u}}|.$$

The proof follows directly from Def. 1.

Proof. Since there exists no solution to Prob. 2, it is ensured that there exists one Ω which begins at the transition state $\mathbf{x}_t$ (see Sec. IV-B). Thus, the state $\mathbf{x}_t$ is always inside a PCIS regardless of the configuration of $\mathcal{B}^{\mathrm{u}}$. □

In order to apply Prop. 1 it is necessary to ensure that the set $\mathcal{B}^{\mathrm{u}}$ cannot cover all PCISs. In [22] it was shown that the SCP is a NP-complete problem. However, there exists a variety of approximation algorithms for this problem [23]. Recent inapproximability results show that the greedy algorithm is the best-possible polynomial time approximation algorithm [24]. Therefore, in any step the subset that covers the largest number of elements not covered yet must be determined.

V. Implementation

In this section we present an example implementation of the approach presented in Sec. IV which is suitable for fast flying helicopters. The focus lies on a special construction of loiter trajectories (PCISs) which together with the parametric model of the obstacles allow for an efficient approximation of the set cover problem Prob. 2. This in turn is used to ensure feasibility of the robot according to Prop. 1. Since overhead power lines are a source for a high potential of hazard for helicopters, we focus on conductors for this implementation. In the following subsections we describe the construction of a certain set of PCIS allowing to approximate the worst case configuration (number of covered PCISs) of power lines.

A. Generation of Connected PCIS*s*

Similar to the approach described in [6] we sample the inputs of the helicopter model in order to retrieve dynamic feasible loiter maneuvers. The set of connected PCISs $\{\Omega_1, \ldots, \Omega_{\mathrm{N}_\Omega}\}$ is represented by a set of independent loiter circles and a set of transition trajectories sharing the same initial state $\mathbf{x}_t$ (transition state) and end in one of the loiter circles. Formally, the transition trajectories also belong to the PCIS, however, the separation is later used to allow for an efficient approximation of the set cover problem. Additionally, we require, that the transition trajectories lie completely inside the known workspace $\mathcal{W}^{\mathrm{k}}$ at time t_0. Furthermore, the considered PCISs are loiter trajectories which form a set of concentric circles along the z-axis. Each level contains a set of concentric circles which all have the same distance $d_{\min}$ to each other. The same distance applies also between circles of the different levels. An example set of loiter circles is shown in Fig. 3. As illustrated, it is required that the loiter circles have the same minimum distance to each other in the xy-plane and along the z-axis.

For the generation of the transition trajectories and the loiter trajectories we apply a slightly modified version of the algorithm described in [6]. This approach samples the input space of the helicopter model to generate a set of possible PCIS which end in a loiter circle for high velocities. In order to generate loiter circles which have constant distance in the xy-plane and in the z-axis we separate the problem. For the loiter circles in the xy plane, we generate multiple trajectories by adopting the roll rate $\dot{\theta}$ of the robot ensuring that the resulting banked turns result in a radius r_k with a constant offset to the radius of the reference loiter circle r_i,

$$r_i = \frac{\|\boldsymbol{v}\|^2}{g\tan(\theta)}, \quad \|r_i - r_k\| = k\ d_{\min},$$

where k is an integer and g is the standard gravity. The radii must be increased or decreased depending if the reference trajectory is turning right or left. This ensures that the position of the loiter trajectories are parallel to each other and have the distance $d_{\min}$.

In order to generate the loiter circles in different levels along the z-axis, a bang-bang control signal for the acceleration in z-direction is applied. This ensures that we use the maximum dynamic capabilities of the robot in the z-dimension and reach the maximum possible deviation in z-direction. Furthermore, it must be ensured that the loiter circles are parallel to the xy-plane which is equivalent with a velocity of zero in z-direction. For the robustness analysis it is required that all loiter circles and the transition trajectories lie inside of $\mathcal{W}^{\mathrm{k}}$ when the robot reaches the transition state.

B. Robustness of Connected PCISs

In order to ensure feasibility of the set of connected PCISs, one has to show that there exists no configuration of $\mathcal{B}^{\mathrm{u}}$ which can cover all PCISs. If any occupancy of the obstacles intersect with the occupancy of the robot while navigating along the PCIS, the PCIS is covered. Since we separated the transition trajectories (Sec. V-A) from the loiter circles, we only have to reason about the unknown obstacles $\mathcal{B}^{\mathrm{u}}$ for the loiter circles. In the next subsection the parametric

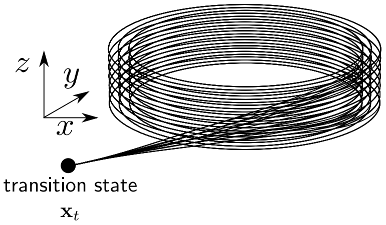

Fig. 3: Figure depicts a set of loiter trajectories with 7 different levels in z-direction and 3 loiter circles per level. All loiter circles have the same distance of $d_{\min}$ to each other.

model for the conductor (power line) is presented.

1) Overhead Power Line: The shape of a conductor can be represented by its catenary curve [25]

$$c(x) = z = \frac{H}{m_C g} \cosh\Big(\frac{m_C g x}{H}\Big)$$

where w.l.o.g. the conductor is aligned with the x-axis and m_C is the conductor mass per length, H is the horizontal component of the conductor tensile force and g is the standard gravity. The factor $\frac{H}{m_C g}$ is called the parameter of the catenary curve. The slope of the conductor is given as

$$c'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \sinh\Big(\frac{m_C g x}{H}\Big).$$

Since there are a lot of different kind of conductors ($m_C \in [60, 3000]^{\mathrm{kg}}/_{\mathrm{km}}$ [26]) which are mounted to transmission towers which varying spacing, we consider intervals for the parameters of the conductor model. The maximum slope of the conductor is at the pole with the widest spacing and with the highest mass. The profile size of the conductor is neglected and it covers a loiter circle Ω if it intersects with the occupancy of the robot

$$\exists \mathbf{x} \in \Omega, \exists [x_c, y_c, z_c] \in \mathcal{A}(\mathbf{x}),$$

where $[x_c, y_c, z_c]$ represent a valid point along the conductor curve.

C. Robustness

In this section we discuss the robustness of the set of connected PCISs regarding the set of obstacles $\mathcal{B}^{\mathrm{u}}$. In order to approximate the set cover problem, one need to determine which obstacle configuration can cover most of the loiter circles, which are not yet covered. This problem requires to find the worst case configuration of the conductor regarding the given set of loiter circles. For the presented conductor model this requires to solve a 6 dimensional problem: start position of conductor, mass, orientation in the xy plane and span length.

In order to identify the maximum number of loiter circles a conductor can cover, we take a perpendicular 2D slice of the loiter pattern relative to the motion direction of the robot. This slice of the robot occupancy along the loiter patterns results in a set of evenly spaced circles $\mathcal{A}^{2d}$, assuming the occupancy of the robot is represented by a sphere. An example set of loiter circles with a possible slice is shown in Fig. 4. This allows to transform the set cover problem from a 3D workspace to a 2D workspace with circles.

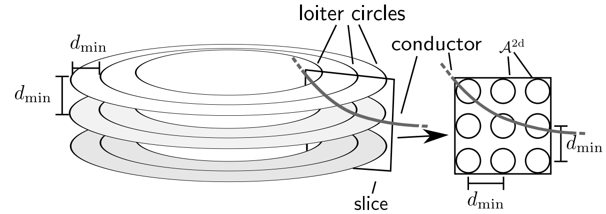

Fig. 4: The different loiter maneuvers have the same minimum distance of $d_{\min}$. In order to perform the robustness analysis, a slice which is perpendicular to the motion direction of the robot is generated.

Proposition 2 (Maximum Number of Circles Covered)**.** *If the slope of the conductor exceeds the slope of a line which is tangential to two vertical circles, there is a chance that the conductor can cover more than one circle along the z-axis. If the shape of the conductor fulfills the following conservative criteria it covers k times 2 vertical loiters:*

$$c(\tfrac{l}{2}) - c(\tfrac{l}{2} - a) \geq (k+1)(d_{\min} - 2r) + (k-1)4r \quad (1)$$

with

$$a = (k-1)d_{\min} + 2r$$

The total length of the conductor is l and r describes the radius of the circle resulting from the slice of the robot occupancy $\mathcal{A}$ at the loiter circles. Assuming that the width of the 2D slice is shorter than $\frac{l}{2}$ and a conductor cannot cover more than 2 vertical loiters in one column, the maximum number of loiter circles covered by one conductor $\mathrm{N_c}$ is given as

$$\mathrm{N_c} = k + \mathrm{N_l^{xy}}.$$

Where $\mathrm{N_l^{xy}}$ is the number of loiter circles in one level (xy plane).

It is noted, that we do not consider conductors which can cover more than 2 vertical loiters, but the generalization for this criteria is straight forward.

Proof. Depending on the distance $d_{\min}$ between the circles and their radius r, there exists a minimum slope τ_s that ensures the existence of a line which is tangential to two vertical circles. The maximum slope of the catenary curve must have at least the value of τ_s such that it can cover two vertical circles of one column. If the catenary curve covers multiple vertical circles, the height difference (z-direction) for a given distance in x-direction needs to exceed a certain value. This geometric criteria for a catenary curve with a bigger slope than τ_s is sketched in Fig. 5 for $k = 2$. Since a conductor of length l has the absolute maximum slope at the two end points of $-\frac{l}{2}$ or $\frac{l}{2}$ (and it is a monotonic function between $[0, \frac{l}{2}]$), it is sufficient to examine this criteria at one of the end points of the catenary curve. Independent of the slope of the catenary curve, it can also cover one circle in each column. Thus, the sum of the number of columns $\mathrm{N_l^{xy}}$ and the number of multiple vertical covers k result in a conservative estimate for the maximum number of circles covered by one catenary curve. □

Finally, we show that the worst case orientation of a conductor is perpendicular to the motion direction of the

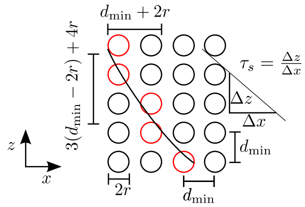

Fig. 5: The criteria from Prop. 2 for covering vertical loiters for $k = 2$ is depicted. Covered circles are red and uncovered are black.

robot.

Proposition 3 (Worst Case Conductor Orientation)**.** *The worst case orientation of the conductor is always perpendicular to the motion direction of the robot along the loiter circles.*

Proof. The perpendicular 2D slice results in circles with the minimum possible distance to each other $d_{\min}$. For the same catenary curve, a smaller value of $d_{\min}$ would result in a higher number of k according to (1). Thus, the perpendicular slice constitutes the worst case orientation of the conductor. □

VI. SIMULATION

In this section simulation scenarios are used to evaluate the implementation from Sec. V. For the simulation scenarios, the helicopter model described in [6] is applied which is based on the fixed-wing model. The maximum roll rate is $\dot{\theta}_{\max} = 0.5\,\mathrm{rad/s}$, the maximum roll angle is $\theta = 1.0\,\mathrm{rad}$ and the maximum acceleration in z-direction is $1.0\,\mathrm{m/s^2}$ with a maximum velocity in z-direction of $5\,\mathrm{m/s}$.

A. Worst Case Loiter Cover

In this section we apply the approach described in Sec. IV-C in order to determine the maximum number of loiter circles a conductor can cover. Therefore, we generated a set of loiter circles with a distance of $\mathrm{d_{min}} = 8\,\mathrm{m}$ and generate 3 circles in each of the 5 levels along the z-axis resulting in 15 loiter circles. The robot occupancy is represented by a sphere with radius $2\,\mathrm{m}$. The minimum slope necessary to cover two vertical circles is $\tau_s = 1.73$. For the conductor model, we consider the interval of masses per length of $60 \to 1000\,\mathrm{kg/km}$ and span range of $50 \to 800\,\mathrm{m}$. Thus the maximum slope of the conductor is 1.8221. According to Prop. 2 this results in a worst case cover of 4 loiter circles. Given the connected set of loiter circles, we can conservatively approximate the set cover problem by

$$\mathrm{N_s} = \left\lfloor \frac{15}{4} \right\rfloor = 3,$$

where $\lfloor\cdot\rfloor$ is the floor function. This solution is conservative, since it assumes that for each conductor an independent worst case configuration exists. A possible configuration for one example worst case configuration of a conductor is shown in Fig. 6.

B. Maximum Feasible Velocity

In order to evaluate the performance gain for the novel approach, we determine the maximum allowed velocity for

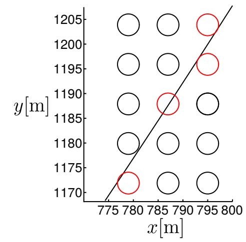

Fig. 6: The conductor covers 4 loiter circles (red circles) which is one possible worst case configuration for this loiter pattern.

straight level flight. Therefore, the robot performs a constant velocity trajectory for $4\,\mathrm{s}$ and then safety for the final state must be ensured. We determine the maximum allowed velocity for a sensor range of $180\,\mathrm{m} \to 575\,\mathrm{m}$, whereby the opening angle of the sensor in the xy plane is π and $\pi/2$ along the z-axis.

The maximum feasible velocity is calculated by the three different approaches: stopping distance approach, single emergency maneuver approach and the novel connected emergency maneuver approach. The braking distance approach determines the fastest speed which still allows to come to a hover inside the current $\mathcal{W}^{\mathrm{k}}(t_0)$ of the robot. The emergency maneuver approach is less conservative and determines the maximum feasible velocity by determining a loiter maneuver, combination of braking and roll maneuver, which lies inside the current $\mathcal{W}^{\mathrm{k}}(t_0)$. Both approaches are described in [6]. The results of all three approaches are depicted in Fig. 7.

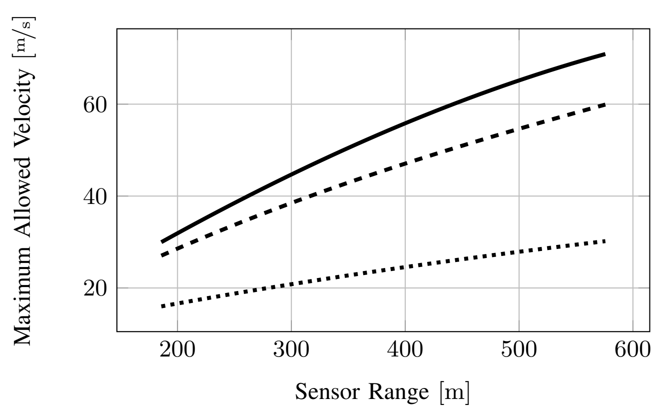

Fig. 7: Allowed maximum velocity relative to the available sensor range is shown. The solid line shows the novel approach based on connected PCISs, the dashed line shows the result for the single emergency maneuver approach [6] and the dotted line shows the braking distance approach.

The novel approach outperforms both other approaches for all considered sensor ranges. Compared to the single emergency maneuver approach, the minimum relative difference is $10.86\,\%$ and the maximum relative difference is $18.37\,\%$.

The resulting loiter circles for the emergency maneuver and for the connected emergency maneuver approach are shown in Fig. 8 for the velocity $65\,\mathrm{m/s}$. From the transition state $\mathbf{x}_t$ till the border of $\mathcal{W}^{\mathrm{k}}(t_0)$ the robot has $4.16\,\mathrm{s}$ allowing to reach 15 loiter circles $\{\Omega_1, \ldots, \Omega_{15}\}$. The robot can reach 5 different levels along the z-axis and 3 loiter in

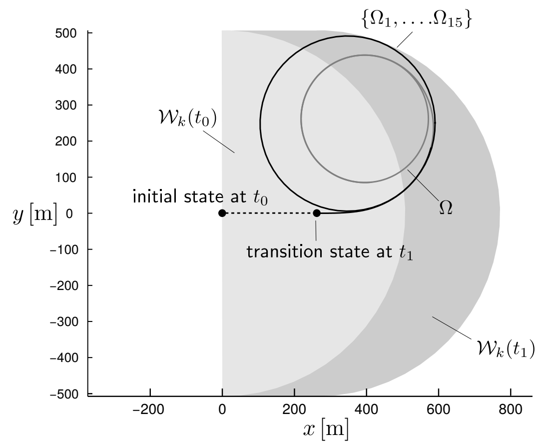

Fig. 8: Figure shows comparison between the invalid single emergency maneuver and the connected set of emergency maneuvers at $65\,\mathrm{m/s}$. The connected emergency maneuvers are depicted by a solid black line and the single emergency maneuver is shown as a dark gray line. The different known spaces $\mathcal{W}^{\mathrm{k}}$ are shown at the initial time and at the transition state. Due to the scale, only one maneuver is shown for the connected emergency maneuvers. The dashed line shows the intended trajectory of the robot.

each level with a minimum distance of $\mathrm{d_{min}} = 8\,\mathrm{m}$ between the loiter circles. As determined in Sec. VI-A, feasibility is still ensured for 3 conductors placed anywhere in $\mathcal{W}^{\mathrm{u}}$.

VII. Conclusions and Future Work

In this paper we discuss the problem of safety for a robotic system in partially-known environments. To the best of the authors' knowledge, this is the first time that the robustness of positive control invariant sets is discussed regarding unknown obstacles in the environment. We have shown that feasibility of the robot in partially-known environments can be guaranteed if the finite set of unknown obstacles cannot block all positive control invariant sets of the robot. Therefore, the worst case configuration of the unknown obstacles is formulated as a set cover problem. An on-line capable implementation for loiter circles is presented. Furthermore, simulation scenarios are presented verifying the novel concept and the implementation. It is shown that the novel approach is less conservative than previous known approaches and increases the maximum allowed velocity up to $18\,\%$.

In future work we want to integrate more classes of obstacles and arbitrary loiter trajectories. Furthermore, we want to apply a more enhanced model of the perception system, such that obstacles have a different detection distance depending on parameters such as size and orientation of the obstacle. Additionally, fight tests with the Boeing Unmanned Little Bird are scheduled in order to conduct an experimental evaluation.

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