Minimum k-Connectivity Maintenance for Robust Multi-Robot Systems

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***Abstract*— In many multi-robot applications, it is critical to maintain connectivity within the robotic team to allow for information exchange and coordination. While most of the existing works focus on connectivity control that ensures robotic team remain connected as one component without faults, we consider the problem of robust connectivity maintenance that seeks to maintain $k-$*connectivity*, such that the multi-robot network could stay connected with the removal of fewer than k robots. In this paper, we propose provably minimum $k-$connectivity maintenance algorithms for multi-robot systems. This ensures the robustness of the multi-robot network connectivity at all time and also in a flexible and optimal way to provide the highest freedom for robots task-related controllers. Particularly, we propose a $k-$Connected Minimum Constraints Subgraph (k-CMCS) algorithm that activates the minimum $k-$connectivity constraints to the original controllers, and then revise the original controllers in a *minimally invasive fashion*. We demonstrate the effectiveness of our approach via simulations of up to 40 robots in the presence of multiple behaviors.**

I. Introduction

Multi-robot systems have been widely studied for extending its capability of doing complex tasks through cooperative behaviors in a number of applications, such as search and rescue [1], cooperative sensor coverage [2], and environmental exploration [3]. The ability of collaboration in multi-robot systems often relies on the local information sharing and interaction among networked robot members through connected communication graph. As robots are often assumed to interact in a proximity-limited manner due to limited communication range [4]–[7], it is necessary to consider connectivity maintenance that ensures robots stay connected by constraining inter-robot distance while executing original tasks. Moreover, it is critical to consider the robustness of the multi-robot network as the expected number of robot failures could grow along with the increasing number of robots. Even for robotic swarm applications with consensus-based controllers that are robust of loss of robots, e.g. flocking [8], [9], maintaining certain level of connectivity could keep robots from converging to incorrect consensus and help improve the convergence of consensus-based controllers.

For many multi-robot applications with parallel tasks, performing multiple and possibly conflicting task-prescribed behaviors simultaneously could easily lead to communication disconnection in robot teams. In such vulnerable scenario, the multi-robot network is more sensitive to faulty situation as well, where a single robot failure could disconnect the entire robotic group. Thus, it demands for fault-tolerant connectivity maintenance to increase the robustness of multi-robot systems. The mixing of parallel multi-robot behaviors with mentioned robust connectivity constraints is particularly challenging for existing work since (a) the additional connectivity maintenance brings increased complexity for global connectivity control algorithms [5], [10], [11] due to the discontinuity from dynamic topology changes as pointed out in [6], and (b) there are no theoretical guarantee on the optimality of imposed connectivity constraints, e.g. [4], [7], [12], [13] nor the perturbation to the original behavior-prescribed controllers due to the constraints, e.g. [5], [14]–[16]. Such issues could lead to overly conservative robots motion and thus behavior failure, for example, dead locks that might prevent the desired execution of behaviors, and inefficiency incurred by the perturbation of connectivity on control outputs between different behaviour groups. Hence, it is desired to derive an approach to maintain minimum satisfying connectivity so as to provide highest freedom for robots' original controllers while ensuring robustness of the multi-robot network.

The objective of this paper is thus to develop provably optimal algorithms for robust but flexible multi-robot connectivity control, by proposing minimum $k-$connectivity maintenance methods that achieve global redundant network connectivity while enabling the robot team to perform various behaviors *at best*. Note that we are not optimizing multi-robot task allocation that determines how to assign different behavior controllers to the robots. We assume the behavior allocation has been done and each robot already knows its real-time behavior-prescribed controller before revising it to accommodate the connectivity and collision avoidance constraints. To formulate the constrained multi-robot control problem, we employ control barrier functions [12], [17] that characterize and enforce safety and connectivity constraints over multi-robot controllers in an optimal way. However, the existing control barrier functions on connectivity requires either predefined fixed connectivity topology [4] or enumerating all possible combination of connectivity topology [12]. Such rigid constraints is not scalable nor feasible in robust connectivity maintenance as number of robots and behaviors increases. To that end, we propose to achieve provably optimal robust connectivity maintenance by developing: 1) a novel quantifiable relationship between original behavior-prescribed controllers and the candidate connectivity constraints, 2) a novel $k-$*Connected Minimum Constraints Subgraph ($k-$CMCS)* method to activate dynamic quantified minimum connectivity constraints, which are *least violated* by the original unrevised behavior-prescribed controllers, and

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3) a unified optimization framework to revise the robots controllers in presence of activated connectivity and collision avoidance constraints that are minimally invasive to the original behavior-prescribed controllers. This enables the multi-robot systems to execute different behaviors simultaneously on a single connected robot team with required robust connectivity. We will first give the formal definition of the constrained $k-$connectivity maintenance problem, and introduce our algorithm of constructing the $k-$Connected Minimum Constraints Subgraph in an *optimal fashion.* Further, we will present our results in evaluating metrics of our $k-$CMCS methods as well as performance of up to 40 robots in simulation.

Our paper presents the following contributions: (1) a generalized $k-$connectivity maintenance framework to enable efficient execution of different behaviors *at best* within a single robot team, while ensuring optimal global robust connectivity and collision avoidance; (2) a novel $k-$CMCS method with quantified relationship between behavior controllers and $k-$connectivity constraints is proposed to efficiently invoke minimum $k-$connectivity constraints with provably optimality guarantee, which could minimally revise any behavior-prescribed controllers.

II. Related Work

The general problem of connectivity maintenance has been widely studied in the past decade due to its importance in enabling local information sharing and collaboration for multi-robot systems in performing complex tasks. Given an initially connected multi-robot spatial communication graph, the goal of continual connectivity control is to couple the task-related controllers of robots with connectivity controller such that the communication graph over time remains connected. There have been two major classes of connectivity control methods: 1) *local methods* that seeks to preserve the initial connectivity graph topology over time [8], [18], [19], and 2) *global methods* that aims to preserve the global algebraic connectivity of the communication graph by deriving controllers to keep the second smallest eigenvalue of the graph Lapacian positive at all time [5], [6], [10], [11], [14]. While the global connectivity control provide better flexibility over local methods as it allows for changing network topology, both of the methodologies demands for the revision of original robot controllers more or less at all time, even if the robots' original behaviors won't lead to network disconnection. This could overly constrain robots' motion when extending to redundant $k-$connectivity maintenance.

To achieve more flexible connectivity control with multiple behaviors, i.e. simultaneously exploring different regions, recent work [15], [16], [20], [21] have explored the idea of redeploying a certain number of robots to act as communication relays, while aiming to allow the rest of the robots to perform their original tasks. In particular, the communication relays can be derived by following certain structured behaviors such as lattice-based formations [20], [21], or by separate optimization process that explicitly assigns some of the robots as connectors [15], [16]. In order to find a more flexible communication relay structure with quantified pairwise connectivity, [22] proposed to employ minimum spanning tree topology and uses pairwise distance as heuristic to provide better freedom of robot motion, i.e. robots closer to each other are less restrictive. However, these heuristic methods have no theoretical guarantee that the selected connectivity constraints are minimum to the original task-related robot controllers.

The problem of $k-$connectivity control or $k-$redundancy control has also been studied [23]–[26]. [25] introduced distributed algorithms for detecting $k-$connectivity of multi-robot graph. Work in [23] addressed the $k-$hop connectivity control where the robots stay connected with its $k-$hop neighbors at all time. In [24], [26] the robots are tasked to reconfigure their positions for meeting certain redundant connectivity constraints. These approaches often consider the connectivity maintenance as a separate optimization problem and hence has no optimal guarantee over the original robot's controllers. For less restrictive multi-robot control with constraint satisfaction, control barrier functions have been employed to encode a variety of inter-robot constraints and the resulting constrained control outputs lead to forward invariance of the satisfying set, i.e. robots remain collision free and connected under predefined fixed communication topology [4], [12], [13]. Although the resultant control outputs are optimized to stay as close to the original controllers with constraints, the predefined fixed communication topology has no guarantee regarding its optimality to the robot behaviors. In our work, we are optimizing both the activated $k-$connectivity constraints together with the controllers with proven optimality guarantees, so that the control revision with the invoked connectivity constraints is minimally invasive to the original behavior-prescribed controllers, allowing for flexible various mutli-robot behaviors *at best*.

III. Problem Formulation

Consider a heterogeneous robotic team $\mathcal{S}$ consisting of n mobile robots in a planar space, with the position and single integrator dynamics of each robot $i \in \{1, \ldots, n\}$ denoted by $x_i \in \mathbb{R}^2$ and $\dot{x}_i = u_i \in \mathbb{R}^2$ respectively. Each robot can connect and communicate directly with other robots within its spatial proximity. The communication graph of the robotic team is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each node $v \in \mathcal{V}$ represents a robot. If the spatial distance between robot $v_i \in \mathcal{V}$ and robot $v_j \in \mathcal{V}$ is less or equal to the communication radius R_c (i.e. $\|x_i - x_j\| \leq R_c$), then we assume the two can communicate and edge $(v_i, v_j) \in \mathcal{E}$ is undirected (i.e. $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$).

We assume the robotic team has been tasked with m simultaneous behaviors, partitioning the set of robots into m sub-groups. To simplify our discussion, we assume the sub-group partitions and behavior controllers are given or already derived from other multi-robot task allocation algorithms, namely, each robot i has been assigned to a sub-group with some behavior-prescribed controller $u_i = \hat{u}_i$. To ensure successful multi-robot coordination and information exchange, it is required that the communication/connectivity graph $\mathcal{G}$ is connected. Moreover, in presence of possible robots failure, the graph should be *robust* in that the removal of certain number of robot nodes won't disconnect the connectivity graph for the remaining robot team, which leads to the following definition of *k-node connected graph* [27].

Definition 1. *(k-node connected graph) A connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be k-node connected (or k-connected) if it has more than k nodes and remains connected whenever fewer than k nodes are removed.*

Given a desired number of k due to robustness requirements on robots failure, we first assume the current multi-robot graph $\mathcal{G}$ is already $k-$connected (for the rest of the paper, $k-$connected refer to $k-$node connected). Then we would like to enforce such constraint as robots execute their behavior-prescribed controllers, so that the resulting time-varying connectivity graph $\mathcal{G}$ is $k-$connected at all time. In presence of the above connectivity constraints as well as the physical constraints of the robots such as inter-robot collision avoidance and velocity limits, each robot i may have to modify their primary task-related controller $\hat{u}_i$ to accommodate the constraints. To that end, the objective is to 1) coordinately invoke active constraints to follow (particularly the connectivity constraints imposed between pair-wise robots), such that the modification to the primary controller is minimum for the robotic team, and 2) compute the modified controllers for robots task execution. In the remaining of this section, we will discuss the formulation of the mentioned constraints in the form of Barrier Certificates on the controllers and will present the formalized optimization problem.

A. Safety and Connectivity Constraints using Barrier Certificates

During movements of multi-robot systems, the robots should avoid collisions with each other to remain safe. Consider the joint robot states $\mathbf{x} = \{x_1, \ldots, x_n\} \in \mathbb{R}^{2n}$ and define the minimum inter-robot safe distance as R_s, for any pair-wise inter-robot collision avoidance constraint between robots i and j. We have the following condition defining the safe set of $\mathbf{x}$.

$$\begin{aligned} h^s_{i,j}(\mathbf{x}) &= \|x_i - x_j\|^2 - R_s^2, \qquad \forall i > j \\ \mathcal{H}^s_{i,j} &= \{\mathbf{x} \in \mathbb{R}^{2n} : h^s_{i,j}(\mathbf{x}) \geq 0\} \end{aligned} \tag{1}$$

The set of $\mathcal{H}^s_{i,j}$ indicates the safety set from which robot i and j will never collide. For the entire robotic team, the safety set can be composed as follows.

$$\mathcal{H}^s = \bigcap_{\{v_i, v_j \in \mathcal{V} : i > j\}} \mathcal{H}^s_{i,j} \tag{2}$$

[17] proposed the safety barrier certificates $\mathcal{B}^s(\mathbf{x})$ that map the constrained safety set (2) of $\mathbf{x}$ to the admissible joint control space $\mathbf{u} \in \mathbb{R}^{2n}$. The result is summarized as follows.

$$\mathcal{B}^s(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^{2n} : \dot{h}^s_{i,j}(\mathbf{x}) + \gamma h^s_{i,j}(\mathbf{x}) \geq 0, \forall i > j\} \tag{3}$$

where γ is a user-defined parameter to confine the available sets. It is proven in [17] that the forward invariance of the safety set $\mathcal{H}^s$ is ensured as long as the joint control input $\mathbf{u}$ stays in set $\mathcal{B}^s(\mathbf{x})$. In other words, the robots will always stay safe if they are initially inter-robot collision free and the control input lies in the set $\mathcal{B}^s(\mathbf{x})$. Note that at any time point t with known current robot states $\mathbf{x}(t)$, the constrained control space in (3) corresponds to a class of linear constraints over pair-wise control inputs u_i and u_j for $\forall i > j$. Note that static obstacles may also be modelled in the same manner if treated as robots with zero velocity.

Next, we consider the pair-wise connectivity constraints among the robotic team. If the connectivity constraint is enforced between pair-wise robots i and j to ensure inter-robot distance not larger than communication range R_c, we have the following condition.

$$\begin{aligned} h^c_{i,j}(\mathbf{x}) &= R_c^2 - \|x_i - x_j\|^2 \\ \mathcal{H}^c_{i,j} &= \{\mathbf{x} \in \mathbb{R}^{2n} : h^c_{i,j}(\mathbf{x}) \geq 0\} \end{aligned} \tag{4}$$

The set of $\mathcal{H}^c_{i,j}$ indicates the feasible set on $\mathbf{x}$ from which robot i and j will never lose connectivity. Consider any connectivity spanning subgraph $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c) \subset \mathcal{G}$ to enforce, the corresponding constrained set can be composed as follows.

$$\mathcal{H}^c(\mathcal{G}^c) = \bigcap_{\{v_i, v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}^c\}} \mathcal{H}^c_{i,j} \tag{5}$$

Similar to the safety barrier certificates in (3), the connectivity barrier certificates are defined as follows and indicate another class of linear constraints over pair-wise control inputs u_i and u_j for $(v_i, v_j) \in \mathcal{E}^c$ at any time point t.

$$\mathcal{B}^c(\mathbf{x}, \mathcal{G}^c) = \{\mathbf{u} \in \mathbb{R}^{2n} : \dot{h}^c_{i,j}(\mathbf{x}) + \gamma h^c_{i,j}(\mathbf{x}) \geq 0, \forall (v_i, v_j) \in \mathcal{E}^c\} \tag{6}$$

B. Objective Function

Consider that a task-related primary behavior control input $\hat{u}_i \in \mathbb{R}^2$ has been computed for each robot i before considering the mentioned constraints. The robotic team needs to determine whether and how to best modify its primary control input in a minimally invasive manner so as to achieve task-related behaviors while ensuring safety and $k-$connectivity. With the defined forms of constraints in (3) and (6), we formally define the *minimum $k-$connectivity maintenance* problem with given k at any time point t as follows.

$$\mathbf{u}^* = \arg\min_{\mathcal{G}^c, \mathbf{u}} \sum_{i=1}^{n} \|u_i - \hat{u}_i\|^2 \tag{7}$$

$$\text{s.t.} \quad \mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c) \subseteq \mathcal{G} \quad \text{is } k-\text{connected} \tag{8}$$

$$\mathbf{u} \in \mathcal{B}^s(\mathbf{x}) \bigcap \mathcal{B}^c(\mathbf{x}, \mathcal{G}^c), \quad \|u_i\| \leq \alpha_i, \forall i = 1, \ldots, n \tag{9}$$

The above Quadratic Programming (QP) optimization problem is to find the optimal active connectivity spanning subgraph $\mathcal{G}^c \subset \mathcal{G}$ from current connected multi-robot connectivity graph $\mathcal{G}$ and the alternative control inputs $\mathbf{u}^* \in \mathbb{R}^{2n}$ bounded by maximum velocity α_i for each robot, so that $k-$connectivity, safety and velocity constraints described in (8) and (9) are always guaranteed while ensuring minimally invasive to the primary controller as shown in (7). Note that as information regarding the primary task is not required other than $\hat{u}_i$, the objective of the original controller may not be guaranteed in form of (7) especially when it conflicts with connectivity or safety constraints, e.g. dispersing robots to different goal locations where robots get disconnected due to limited communication range. In this case, the objective of (7) first ensures constraints are satisfied at all time and then minimizes the deviation from original controller, e.g. dispersing robots towards assigned goal locations as much as possible while keeping them safe and k-connected.

Most of the works focus on the $1-$connectivity maintenance [5], [6], [8], [11], [12] and are intractable to apply to $k-$connectivity due to lack of optimality from the invoked connectivity constraints. In particular, many of them solve the connectivity control problem with either predefined static connectivity topology constraints or enumerate all compositions of various connectivity topology. These approaches (a) are not capable to deal with more constrained $k-$connectivity problem, where the connectivity constraints need to be generated in a minimally invasive fashion in favor of current behaviors, (b) are not scalable to large numbers of robots, as they need to enumerate all static connectivity constraints.

In this paper, we propose to decouple the $k-$connectivity maintenance problem into two dependent sub-problems, namely 1) select provable optimal $k-$connected subgraph $\mathcal{G}^c = \mathcal{G}_k^* \subseteq \mathcal{G}$ that invokes minimally invasive connectivity constraints over multi-robot behaviors, and then 2) solve the optimization problem (7) with the obtained optimal $k-$connected graph $\mathcal{G}_k^*$. Such a solution is *the first of its kind* in that it enables best flexibility of multi-robot behaviors with provably minimum $k-$connectivity constraints and minimally invasive controller revision in a unified manner.

IV. Maintaining Minimum $k-$Connectivity

A. Min-Size $k-$Node Connected Spanning Subgraph ($k-$NCSS)

First we consider the sub-problem of selecting optimal $k-$connectivity spanning subgraph $\mathcal{G}^{c*} = \mathcal{G}_k^*(\mathcal{V}, \mathcal{E}_k^*) \subseteq \mathcal{G}$ in (7) that introduces minimum $k-$connectivity constraints. Recall that each edge $(v_i, v_j) \in \mathcal{E}^c$ in a candidate graph $\mathcal{G}^c$ enforces one pair-wise linear constraint over primary control inputs $\hat{u}_i$ and $\hat{u}_j$ for robot i and j, as shown in (4). Then the graph $\mathcal{G}^c$ whose edges define the minimum connectivity constraints must exist among the set of all minimum $k-$connected spanning subgraphs from current connectivity graph $\mathcal{G}$. These minimum $k-$connected spanning subgraphs cover all the vertices $\mathcal{V}$ with minimum number of $k-$node connected edges.

Finding such a min-size k-Node Connected Spanning Subgraph has been known as NP-hard for even $k = 2$ [28] and in graph theory there exists a heuristic algorithmic framework, $k-$Node Connected Spanning Subgraph ($k-$NCSS) [28], [29] that finds the approximate min-size k-connected subgraph with uniform edge cost. Briefly, given an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and k, the min-size $k-$connected spanning subgraph $\mathcal{G}_k^*$ can be found by the following summarized algorithm.

Algorithm 1 Minimum-size $k-$node connected spanning subgraph ($k-$NCSS)

Input: $\mathcal{G}(\mathcal{V}, \mathcal{E})$, k
Output: $\mathcal{G}_k^*$
1: find a min-size $k-1$ edge cover $M \leftarrow \arg\min\{|M| : \deg_M(v) \geq k-1, \forall v \in \mathcal{V}, M \subseteq \mathcal{E}\}$
2: find an inclusionwise minimal edge set $F \subseteq \mathcal{E} \setminus M$ such that $(\mathcal{V}, M \cup F)$ is $k-$connected
3: **return** $\mathcal{G}_k^* \leftarrow (\mathcal{V}, M \cup F)$

With the Algorithm 1, we have the following Lemma regarding its known approximation of the derived $k-$connected spanning subgraph $\mathcal{G}_k^*$.

Lemma 2. *([28], [29]) Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a graph of node connectivity $\geq k$. Then the Algorithm 1 finds a $k-$node connected spanning subgraph $(\mathcal{V}, M \cup F)$ such that $|M \cup F| \leq (1 + \frac{1}{k})|\mathcal{E}_{opt}|$, where $|\mathcal{E}_{opt}|$ denotes the cardinality of the optimal solution.*

Hence Algorithm 1 provides a bounded solution to find a $k-$NCSS $\mathcal{G}_k^* \subseteq \mathcal{G}$ with minimum number of edges that can be used to define active pairwise connectivity constraints for ensuring $k-$connectivity. However, it should be noted that such solution seeks to find minimum number of edges with uniform edge cost, while in many cases, there might be multiple solutions given uniform edge cost, and connectivity edges between robots should be weighted differently based on their likelihood of being violated due to the task-related controllers. In other words, we need a better heuristic to search for the minimum $k-$connected spanning subgraph with consideration of heterogeneous edge weights that can reflect the minimum violations of the robots primary controllers, especially when there exists multiple $k-$connected subgraphs with the same number of edges. In the next subsection, we will introduce a new heuristic for edge weight assignment and propose a complete solution to find the minimum $k-$connected spanning subgraph as to the robots task-related controllers.

B. $k-$Connected Minimum Constraints Subgraph ($k-$CMCS)

It is reasonable to assume that a smaller number of connectivity edges to maintain will introduce less constraints over the multi-robot systems. However, as mentioned when there exist multiple $k-$NCSS with same number of edges, we need to break ties so as to introduce the truly minimum constraints over robots controllers. Recall that resultant connectivity constraints due to enforced edges are in the form of (6) over the robots' controllers. Thus, to quantify the strength of connectivity constraint by an edge $(v_i, v_j) \in \mathcal{E}$, we introduce the weight assignment defined as follows.

$$w_{i,j} = \dot{h}_{i,j}^c(\mathbf{x}, \hat{u}_i, \hat{u}_j) + \gamma h_{i,j}^c(\mathbf{x}), \forall (v_i, v_j) \in \mathcal{E} \quad (10)$$

Compared to the connectivity constraint in (6), $w_{i,j}$ indicates the violation of the pair-wise connectivity constraint between the two robots, with the higher value of $w_{i,j}$ the less likely the connectivity constraint being violated. To that end, the present connectivity graph $\mathcal{G}$ can be converted to a weighted connectivity graph $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with $w_{i,j} \in \mathcal{W}$. With the new weight assignment in (10), we recall the heuristic Algorithm 1 and in Line 1, redefine the min-size $(k-1)$ edge cover M' by the following.

$$M' = \underset{M' \subseteq \mathcal{E}}{\arg\min} \quad \beta \cdot |M'| - \Sigma_{(v_i,v_j) \in M'} \{w_{i,j}\} \tag{11}$$

where β is a pre-defined parameter and we assume $\beta >> 2 \cdot \Sigma_{\forall w_{i,j} \in \mathcal{W}} |w_{i,j}|$, so that the selected edge cover set M' has minimum number of edges. With the new condition above for finding $(k-1)$ edge cover set M', a new weighted $k-$connected spanning subgraph can be derived as $\hat{\mathcal{G}}_k^* = (\mathcal{V}, \mathcal{E}'_k, \mathcal{W}_k)$ with $\mathcal{E}'_k = M' \cup F' \subseteq \mathcal{E}$, which we formally defined as *$k-$Connected Minimum Constraints Subgraph ($k-$CMCS)*. In particular, we have the following Theorem on bounded cardinality of edge set $\mathcal{E}'_k$ of the $k-$CMCS $\hat{\mathcal{G}}_k^*$.

Theorem 3. *Given weighted undirected graph $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ of node connectivity $\geq k$. Then the Algorithm 1 with redefined condition (11) finds the $k-$CMCS $\hat{\mathcal{G}}_k^* = (\mathcal{V}, \mathcal{E}'_k, \mathcal{W}_k)$ such that $|\mathcal{E}'_k| \leq (1 + \frac{1}{k})|\mathcal{E}_{opt}|$, where $\mathcal{E}_{opt}$ denotes the cardinality of the optimal solution as in **Lemma** 2.*

Proof: We first prove that the solution $\hat{\mathcal{G}}_k^* = (\mathcal{V}, M' \cup F')$ from modified Algorithm 1 with (11) and $\mathcal{G}_k^* = (\mathcal{V}, M \cup F)$ from original Algorithm 1 have the same number of edges. By contradiction, we assume they have different number of edges in M' and M, namely, the following two conditions must be true at the same time.

$$\begin{gathered} \beta \cdot |M'| - \Sigma_{(v_i,v_j) \in M'} \{w_{i,j}\} < \beta \cdot |M| - \Sigma_{(v_i,v_j) \in M} \{w_{i,j}\} \\ |M'| > |M| \end{gathered} \tag{12}$$

Recall that $\beta >> 2 \cdot \Sigma_{\forall w_{i,j} \in \mathcal{W}} |w_{i,j}|$, hence it is straightforward that the two equations contradicts to each other, proving that $|M'| = |M|$. Then since the Step 2 is the same in both of the algorithms, we conclude that $|\mathcal{E}'_k| \leq (1 + \frac{1}{k})|\mathcal{E}_{opt}|$.□

Besides Theorem 3, it is also straightforward from (11) that the obtained $k-$CMCS has not only the minimum number of edges, but also minimum cumulative weights. The $k-$connectivity constraints invoked from $\hat{\mathcal{G}}_k^*$ are thus minimally violated by the current behavior-prescribed robots controllers, which implies the least restriction due to connectivity requirements. Such $\hat{\mathcal{G}}_k^*$ therefore specifies the optimal $k-$connectivity subgraph $\mathcal{G}^{c*} = \hat{\mathcal{G}}_k^* \subseteq \mathcal{G}$ to enforce for the formal optimization problem in (8). Next, we propose a $k-$Connected Minimum Constraints Subgraph ($k-$CMCS) Algorithm in Algorithm 2 that is modified from Algorithm 1 with complete solution and modification due to (11). In the rest of the paper, we use $k-$CMCS interchangeably to refer to the graph or the algorithm.

Algorithm 2 takes as inputs the weighted present multi-robot connectivity graph $\hat{\mathcal{G}}$ by weight assignment (10) and the value of k. From Line 1-3, the min-size $(k-1)$ edge cover M' in (11) is obtained by first solving for its complementary

Algorithm 2 $k-$Connected Minimum Constraints Subgraph ($k-$CMCS)

Input: $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, k
Output: $\hat{\mathcal{G}}_k^*$
1: **for** all $v \in \mathcal{V}$ **do** $b(v) \leftarrow \deg(v) + 1 - k$
2: Get $b-$matching edge set: $\bar{M}' \leftarrow b-$Suitor$(\hat{\mathcal{G}}, b)$
3: $M' \leftarrow \hat{\mathcal{G}} \setminus \bar{M}'$, $F' \leftarrow \emptyset$, $\mathcal{G}_t \leftarrow \hat{\mathcal{G}}$
4: **for** all $e \in \bar{M}'$ **do**
5: $\quad \mathcal{G}'_t \leftarrow$ CreateDigraph($\mathcal{G}_t$, unit capacities)
6: $\quad$ #disjoint_path $\leftarrow$ max_flow$(\mathcal{G}'_t, e_{source}, e_{sink})$
7: $\quad$ **if** #disjoint_path$> k$ **then**
8: $\quad\quad \mathcal{G}_t$.remove(e)
9: $\quad$ **else**
10: $\quad\quad F' \leftarrow F' \cup e$
11: **return** $\hat{\mathcal{G}}_k^* \leftarrow (\mathcal{V}, M' \cup F')$

edge set $\bar{M}'$ with the following condition.

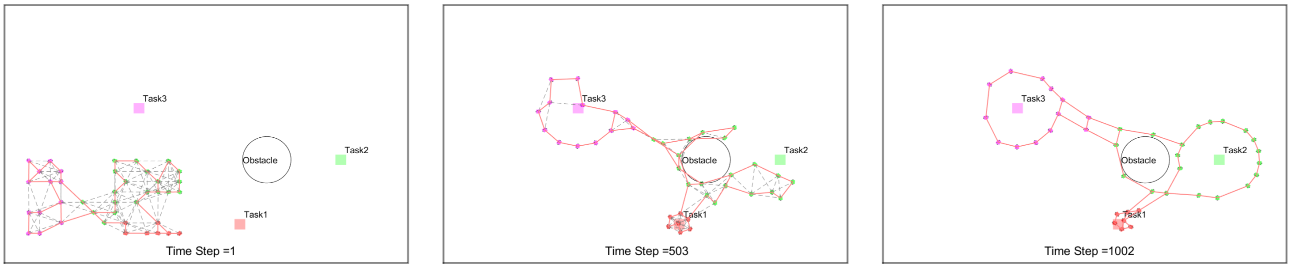
$$\begin{aligned} \bar{M}' = \underset{\bar{M}' \subseteq \mathcal{E}}{\arg\max} \quad & \beta \cdot |M'| - \Sigma_{(v_i,v_j) \in M'} \{w_{i,j}\} \\ \text{s.t.} \quad & \deg_{\bar{M}'}(v) \leq \deg(v) + 1 - k \quad \forall v \in \mathcal{V} \end{aligned} \tag{13}$$

where $\deg(v)$ denotes the degree of node v. The above problem is known as a weighted $b-$matching problem [28], [29] and we implement a subroutine $b-$Suitor [30] to solve it efficiently. When computing for the inclusionwise minimal edge set F' in Line 4-10, we start with empty set F' and initialize the current subgraph to be the present connectivity graph $\hat{\mathcal{G}}$ that is $k-$connected as assumed from the beginning. Then each candidate edge e not in the $k-1$ edge cover set M' is checked by finding if there are at least $(k+1)$ node disjoint paths in the current subgraph $\mathcal{G}$. If yes, then the current candidate edge e is not critical (see [28]) and hence removed from current subgraph. Otherwise, the edge is critical and shall be inserted into the set F' to consist of final $k-$CMCS $\hat{\mathcal{G}}_k^*$. This rely on the fact that for an optimal $k-$connected spanning subgraph with least number of edges, each edge is critical and there will be no more than $k+1$ disjoint path between the two end nodes for the edge [28]. Here we present a solution to efficiently compute the number of disjoint paths by max flow algorithms such as [31] over the current subgraph $\mathcal{G}_t$. The skeleton of Algorithm 2 for finding the $(k-1)$ edge cover set M' and inclusionwise minimal set F' follows the same heuristic as Algorithm 1 and Theorem 3 ensures the quality of our proposed method.

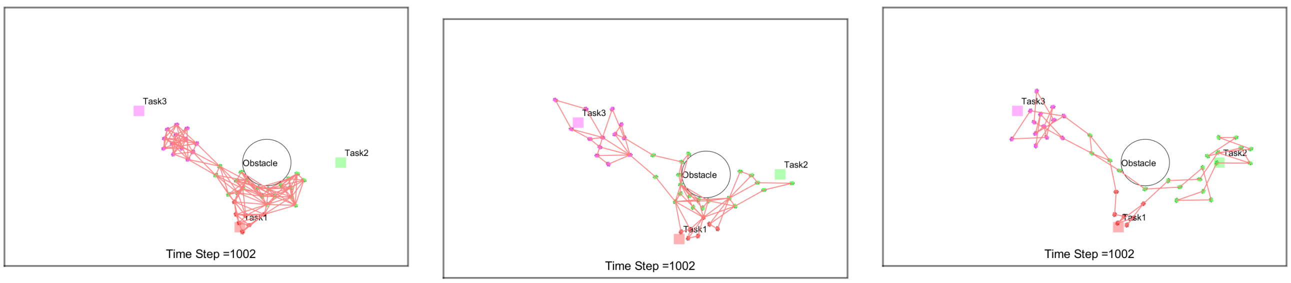
With the final $k-$CMCS $\hat{\mathcal{G}}_k^*$ obtained from our Algorithm 2 as the optimal $k-$connectivity subgraph $\mathcal{G}^{c*} = \hat{\mathcal{G}}_k^*$ in (9), we can specify the safety and connectivity barrier certificates (3) and (6) to invoke linear constraints and efficiently solve the original quadratic programming (QP) problem in (7) to get optimal revised robot controllers satisfying safety and $k-$connectivity constraints with minimum invasion to the original controllers.

V. Results

To evaluate our proposed k-CMCS method for robust connectivity maintenance, we designed two sets of experiments

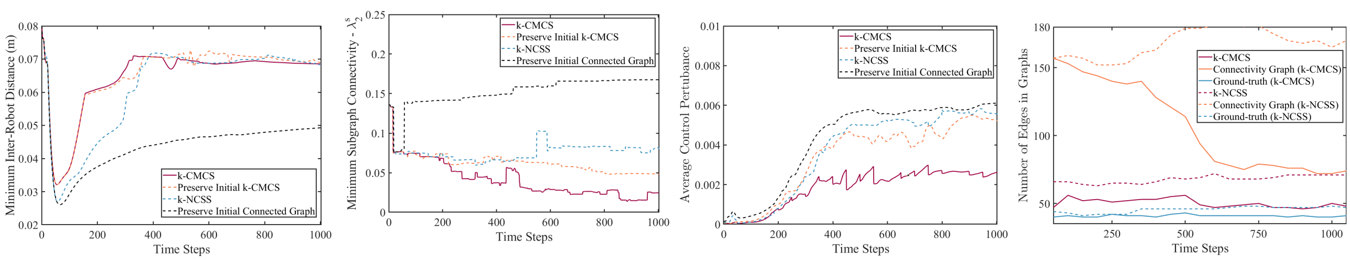


(a) Time Step = 1 (b) Time Step = 503 (k-CMCS) (c) Time Step = 1002 (k-CMCS, Converged)



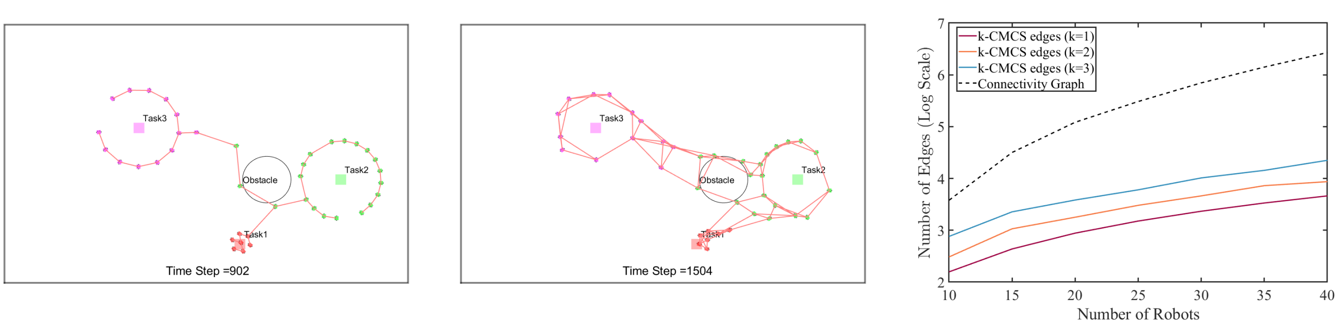
(d) Time Step = 1002 (Fixed Initial Connectivity, Converged) (e) Time Step = 1002 (k-NCSS, Converged) (f) Time Step = 1002 (Fixed Initial k-CMCS, Converged)

Fig. 1: Simulation example of 40 robots tasked to three different places simultaneously with 2-connectivity maintenance ($k = 2$): red robots rendezvous to red task 1 region, while green robots and magenta robots move to region of green task 2 and magenta task 3 and keep orbiting around the regions. Grey dashed lines in (a),(b) denote current connectivity edges and red lines in (a)-(f) denote current active k-connectivity graph invoking pair-wise connectivity constraints. Compared to inter-robot connectivity constraints from (d) initial connectivity graph, (e) minimum k-Node Connected Spanning Subgraph (k-NCSS) [28], [29], and (f) fixed initial k-CMCS (only computes k-CMCS once), the converged result (c) of our proposed k-CMCS approach is able to explicitly demonstrate the three behaviors under robust connectivity constraints due to invoked minimum connectivity constraints on the robots.



(a) Minimum inter-robot distance (b) Minimum Algebraic Connectivity of Subgraph (c) Average control perturbation (d) Number of Edges in Graphs

Fig. 2: Performance comparison of simulation example in Figure 1 w.r.t. different metrics: (a) Minimum inter-robot distance (safety distance is 0.025m), (b) Minimum subgraph algebraic connectivity evaluated by second smallest eigenvalue of laplacian matrix with $k - 1 = 1$ robot being taken out. Positive meaning connectivity ensured. (c) Control perturbation computed by $\frac{1}{n}\sum_{i=1}^{n}\left\|u_i^* - \hat{u}_i\right\|^2$, (d) Number of edges in the corresponding graphs. Note our k-CMCS approach activates less number of k-connectivity edges than k-NCSS [28], [29], and always stay within the ratio of $1 + \frac{1}{k}$ to the groud-truth minimum k-connected subgraph (Benchmarked by brute-force algorithm that exhaustively checks combinations of all edges).



(a) 1-Connectivity with k-CMCS, Converged (b) 3-Connectivity with k-CMCS, Converged (c) Average Number of Edges (Log Scale)

Fig. 3: More results with our k-CMCS method. (a)-(b) are converged results with different k-connectivity requirements. Note when $k = 1$ in (a) the $k-$CMCS degenerates to a minimum weighted spanning tree. (c) Results of number of edges in graphs with $k = 1, 2, 3$ and different number of robots.

in simulation. First, we have $n = 40$ robots already divided into $m = 3$ subgroups simultaneously performing 3 behaviors such as rendezvous to goal and dynamic circle formation behaviors. In particular, we denote the three assigned robot subgroup sets as $\{m_1\}$ (rendezvous-to-goal task 1), $\{m_2\}$ (circle formation task 2), and $\{m_3\}$ (circle formation task 3) with m_1, m_2, m_3 as the number of robot members in each subgroup and $m_1 + m_2 + m_3 = n = 40$. The behavior-prescribed controllers for the three subgroups are as follows.

$$\hat{u}_i = \begin{cases} -\frac{K_p}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}(x_i - x_j) - (x_i - \bar{x}_{task1}),\ \forall i \in \{m_1\} \\ -K_p(x_i - \bar{x}_{i,task2}(t)),\ \forall i \in \{m_2\} \\ -K_p(x_i - \bar{x}_{i,task3}(t)),\ \forall i \in \{m_3\} \end{cases} \tag{14}$$

where

$$\begin{aligned} \bar{x}_{i,task2}(t) &= \bar{x}_{task2} + \begin{bmatrix} R_f \cdot \cos\left(\frac{2\pi}{m_2}\cdot i_{m2} + \Delta\theta\cdot t\right) \\ R_f \cdot \sin\left(\frac{2\pi}{m_2}\cdot i_{m2} + \Delta\theta\cdot t\right) \end{bmatrix}, \\ &\qquad \forall i \in \{m_2\}, \text{and } i_{m2} = 1, \ldots, m_2 \\ \bar{x}_{i,task3}(t) &= \bar{x}_{task3} + \begin{bmatrix} R_f \cdot \cos\left(\frac{2\pi}{m_3}\cdot i_{m3} + \Delta\theta\cdot t\right) \\ R_f \cdot \sin\left(\frac{2\pi}{m_3}\cdot i_{m3} + \Delta\theta\cdot t\right) \end{bmatrix}, \\ &\qquad \forall i \in \{m_3\}, \text{and } i_{m3} = 1, \ldots, m_3 \end{aligned} \tag{15}$$

$\bar{x}_{task1}, \bar{x}_{task2}, \bar{x}_{task3} \in \mathbb{R}^2$ are respectively the goal position of the rendezvous-to-goal behavior task 1 and the two circle formation centers of task 2 and task 3. $K_p > 0$ is the control gain and $\mathcal{N}_i$ specifies neighbors of robot i within the limited communication range R_s. R_f is the orbiting radius of the dynamic circle formation for task 2 and task 3. $\Delta\theta$ specifies the angular increment used to define new target location for each robot at each time step to perform the orbiting circle formation. For the second sets of experiments, we have various number of robots ranging from $n = 10$ to $n = 40$ in the increments of 5 to perform the same set of rendezvous and orbiting behaviors at the same time. With the provided behavior-prescribed controllers, we apply the minimally revised controllers from (7) with single-integrator dynamics to the robots with unicycle dynamics in simulation using kinematics mapping in [13].

The first set of experiments are performed on a team of $n = 40$ mobile robots in presence of a static obstacle as shown in Figure 1, under $2-$connectivity requirement ($k = 2$). With the enforced $2-$connectivity constraints (preserved red edges), taking out no more than $k - 1 = 1$ robots will not disconnect the rest multi-robot connectivity graph. The robot team is divided into $m = 3$ subgroups with different colors and tasked with 3 parallel behaviors. In the figures, robots in red subgroup 1 execute biased rendezvous behaviors towards the red task site 1, while robots in green subgroup 2 and magenta subgroup 3 perform circle formation behaviors around the green task site 3 and magenta task site 4 respectively. Robots update their k-CMCS periodically (every 50 time steps) to invoke the minimum k-connectivity constraints as task progresses. As shown in Figure 1a-c, our k-CMCS approach is able to generate minimum connectivity graph (red edges) from the present connectivity graph (grey edges) so that the invoked connectivity constraints are minimally invasive to the primary behavior controllers. Most of the target behavior configurations have been accomplished as shown in Figure 1c. Particularly, from the results in Figure 1b-c, the communication relays connecting different subgroups are implicitly formed to provide greater flexibility for the rest of the robots while ensuring the $k-$connectivity requirements, without the need of explicit robots roles assignment as done in [15], [16]. This results from the fact that our algorithm enforces provably minimum $k-$connectivity graph that is least restrictive to the robots. As in the intermediate stage captured in Figure 1b, although robots from different subgroups get cluttered around the static obstacle, they are able to generate minimum connectivity graph (red links) from the dense local connectivity graph (grey) so that the team can successfully "unfold" themselves and keep moving with their designated behaviors.

In comparison, we present converged results of other three methods shown in Figure 1d-f: which are i) always preserving initial connected edges (grey edges in Figure 1a) with converged result depicted in Figure 1d, ii) preserving edges in present k-Node Connected Spanning Subgraph (k-NCSS) [28], [29] that seeks to select minimum number of edges without consideration of robot motions (result depicted in Figure 1e), and iii) always preserving edges in initial k-CMCS (red edges in Figure 1a) without updating (result shown in Figure 1f). For results in Figure 1d and f, due to the rigid invoked connectivity graph as the robots move, they can hardly achieve circle formation and could fall into deadlock before reaching the target regions. Without considering the robots' original controllers, k-NCSS method in Figure 1e imposes overly constrained edges even if the number of them are minimum. In contrast, our k-CMCS method selects minimum number of edges and at the same time ensures they are most in favor of the robots original controllers, thus leading to more flexible motions.

Numerical results are also provided in Figure 2 showing our method ensures safety and robust $k-$connectivity, while having minimal control perturbation due to connectivity as compared to other mentioned methods above. In particular, for Figure 2b the minimum subgraph connectivity is obtained by taking out different combinations of $k-1$ robots (1 in this case) on the graphs used to invoke connectivity constraints and output the minimum connectivity after robot removal. It is noted that our k-CMCS is robust to $(k-1)$ robots removal and is least restrictive compared to other compared methods. Moreover, Figure 2d shows that despite of increasing number of existing edges on the original connectivity graph (orange), the number of selected edges by our k-CMCS (solid red) is much smaller and always within the theoretical bound on the ratio of $1 + \frac{1}{k}$ to the ground-truth (solid blue). Thanks to the greater freedom of robots motions allowed from our k-CMCS, the number of edges on the original connectivity graph (solid orange) across time is also significantly reduced compared to the case with k-NCSS method, indicating that the robots in this case separate more quickly to their designated tasks. Lastly, we present the simulation results with different number of robots and various $k-$connectivity requirements in Figure 3. Note that when $k = 1$ the problem reduced to minimum connectivity control and the obtained k-CMCS graph in Figure 3a becomes the minimum spanning tree of the original graph, which has the minimum number of edges required for keeping the connectivity graph connected. Figure 3c indicates that although the number of edges in connectivity graph may grow exponentially as the number of robots increases, the number of our obtained k-CMCS edges will not grow that fast, making the resulting invoked connectivity constraints more scalable.

VI. Conclusion

In this paper, we considered the problem of minimum $k-$connectivity maintenance for flexible multi-robot behaviors. In particular, we proposed a $k-$Connected Minimum Constraints Subgraph ($k-$CMCS) algorithm to compute provably minimum $k-$connectivity constraints as to the robots behavior-prescribed controllers. In this way, the robots controllers will only be revised as necessary in a minimally invasive manner with dynamic and possibly discontinuous communication topology. This algorithm enables simultaneous behaviors at best while maintaining constraints due to collision avoidance and required redundant connectivity that is robust to robots failure. Experimental results validate our method with large number of robots and comparison to other methods, showing the significant improvement on robot team performance on various tasks requiring different behaviors.

Future work includes extensions to fully decentralized k-connectivity control. We will also implement the algorithms on physical robotic platforms to investigate other probabilistic uncertainties in real-world applications to further improve the robustness of the multi-robot systems.

References

[1] G. Kantor, S. Singh, R. Peterson, D. Rus, A. Das, V. Kumar, G. Pereira, and J. Spletzer, "Distributed search and rescue with robot and sensor teams," in *Field and Service Robotics*. Springer, 2003, pp. 529–538.

[2] W. Luo and K. Sycara, "Adaptive sampling and online learning in multi-robot sensor coverage with mixture of gaussian processes," in *IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2018, pp. 6359–6364.

[3] W. Burgard, M. Moors, C. Stachniss, and F. E. Schneider, "Coordinated multi-robot exploration," *IEEE Transactions on robotics*, vol. 21, no. 3, pp. 376–386, 2005.

[4] A. Li, L. Wang, P. Pierpaoli, and M. Egerstedt, "Formally correct composition of coordinated behaviors using control barrier certificates," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2018, pp. 3723–3729.

[5] L. Sabattini, N. Chopra, and C. Secchi, "Decentralized connectivity maintenance for cooperative control of mobile robotic systems," *The International Journal of Robotics Research*, vol. 32, no. 12, pp. 1411–1423, 2013.

[6] R. K. Williams, A. Gasparri, G. S. Sukhatme, and G. Ulivi, "Global connectivity control for spatially interacting multi-robot systems with unicycle kinematics," in *IEEE international conference on robotics and automation (ICRA)*. IEEE, 2015, pp. 1255–1261.

[7] M. M. Zavlanos, M. B. Egerstedt, and G. J. Pappas, "Graph-theoretic connectivity control of mobile robot networks," *Proceedings of the IEEE*, vol. 99, no. 9, pp. 1525–1540, 2011.

[8] M. M. Zavlanos, A. Jadbabaie, and G. J. Pappas, "Flocking while preserving network connectivity," in *IEEE Conference on Decision and Control*. IEEE, 2007, pp. 2919–2924.

[9] K. Saulnier, D. Saldana, A. Prorok, G. J. Pappas, and V. Kumar, "Resilient flocking for mobile robot teams," *IEEE Robotics and Automation letters*, vol. 2, no. 2, pp. 1039–1046, 2017.

[10] L. Sabattini, C. Secchi, N. Chopra, and A. Gasparri, "Distributed control of multirobot systems with global connectivity maintenance," *IEEE Transactions on Robotics*, vol. 29, no. 5, pp. 1326–1332, 2013.

[11] P. Yang, R. A. Freeman, G. J. Gordon, K. M. Lynch, S. S. Srinivasa, and R. Sukthankar, "Decentralized estimation and control of graph connectivity for mobile sensor networks," *Automatica*, vol. 46, no. 2, pp. 390–396, 2010.

[12] L. Wang, A. D. Ames, and M. Egerstedt, "Multi-objective compositions for collision-free connectivity maintenance in teams of mobile robots," in *IEEE Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 2659–2664.

[13] ——, "Safety barrier certificates for collisions-free multirobot systems," *IEEE Transactions on Robotics*, vol. 33, no. 3, pp. 661–674, 2017.

[14] P. R. Giordano, A. Franchi, C. Secchi, and H. H. Bulthoff, "Bilateral teleoperation of groups of uavs with decentralized connectivity maintenance," *Robotics: Science and Systems VII*, p. 273, 2012.

[15] J. Banfi, N. Basilico, and S. Carpin, "Optimal redeployment of multirobot teams for communication maintenance," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2018, pp. 3757–3764.

[16] N. Majcherczyk, A. Jayabalan, G. Beltrame, and C. Pinciroli, "Decentralized connectivity-preserving deployment of large-scale robot swarms," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2018, pp. 4295–4302.

[17] U. Borrmann, L. Wang, A. D. Ames, and M. Egerstedt, "Control barrier certificates for safe swarm behavior," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 68–73, 2015.

[18] D. V. Dimarogonas and K. H. Johansson, "Decentralized connectivity maintenance in mobile networks with bounded inputs," in *IEEE International Conference on Robotics and Automation*. IEEE, 2008, pp. 1507–1512.

[19] M. Ji and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 693–703, 2007.

[20] M. Chandarana, W. Luo, M. Lewis, K. Sycara, and S. Scherer, "Decentralized method for sub-swarm deployment and rejoining," in *IEEE International Conference on Systems, Man, and Cybernetics (SMC)*. IEEE, 2018, pp. 1209–1214.

[21] D. Krupke, M. Ernestus, M. Hemmer, and S. P. Fekete, "Distributed cohesive control for robot swarms: Maintaining good connectivity in the presence of exterior forces," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2015, pp. 413–420.

[22] R. Aragues, C. Sagues, and Y. Mezouar, "Triggered minimum spanning tree for distributed coverage with connectivity maintenance," in *Control Conference (ECC), 2014 European*. IEEE, 2014, pp. 1881–1887.

[23] M. M. Zavlanos and G. J. Pappas, "Controlling connectivity of dynamic graphs," in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 6388–6393.

[24] N. Atay and B. Bayazit, "Mobile wireless sensor network connectivity repair with k-redundancy," in *Algorithmic Foundation of Robotics VIII*. Springer, 2009, pp. 35–49.

[25] A. Cornejo and N. Lynch, "Fault-tolerance through k-connectivity," in *Workshop on network science and systems issues in multi-robot autonomy: ICRA*, vol. 2, 2010, p. 2010.

[26] N. Chakraborty and K. Sycara, "Reconfiguration algorithms for mobile robotic networks," in *IEEE International Conference on Robotics and Automation*. IEEE, 2010, pp. 5484–5489.

[27] R. Diestel, "Graph theory," *Graduate texts in mathematics*, vol. 173, p. 7, 2012.

[28] Z. Nutov, "The k-connected subgraph problem," *Handbook of Approximation Algorithms and Metaheuristics: Contemporary and Emerging Applications, Volume 2*, 2018.

[29] J. Cheriyan and R. Thurimella, "Approximating minimum-size k-connected spanning subgraphs via matching," *SIAM Journal on Computing*, vol. 30, no. 2, pp. 528–560, 2000.

[30] A. Khan, A. Pothen, M. Mostofa Ali Patwary, N. R. Satish, N. Sundaram, F. Manne, M. Halappanavar, and P. Dubey, "Efficient approximation algorithms for weighted b-matching," *SIAM Journal on Scientific Computing*, vol. 38, no. 5, pp. S593–S619, 2016.

[31] Y. Boykov and V. Kolmogorov, "An experimental comparison of min-cut/max-flow algorithms for energy minimization in vision," *IEEE Transactions on Pattern Analysis & Machine Intelligence*, no. 9, pp. 1124–1137, 2004.