Sensor Based Planning for Rod Shaped Robots in Three Dimensions: Piece-wise Retracts of $\mathbb{R}^3 \times S^2$

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Abstract

We describe a new roadmap, termed the rod-HGVG, for motion planning of a rod-shaped robot operating in a three-dimensional volume. This roadmap is defined in terms of work space distance enabling us to prescribe an incremental construction procedure. This allows the robot to explore its configuration space, $\mathbb{R}^3 \times S^2$, without ever explicitly constructing the configuration space. In fact, the rod robot need not know the work space ahead of time. We term the rod-HGVG a piecewise retract because it comprises many retracts. Homotopy theory asserts that there cannot be in general a one-dimensional retract of non-contractible five-dimensional space. Instead, we define an exact cellular decomposition on $\mathbb{R}^3 \times S^2$ and a retract in each cell. Next, we "connect" the retracts of each cell forming a piece-wise retract of the rod's configuration space.

1 Introduction

This work considers the sensor based planning of a rod robot in three dimensions. Although planning for rod-shaped robots is geometrically similar to planning for bimanipulators, ultimately motion planning for highly articulated robots motivates this research and the motion planning of a rod robot is the first step toward this goal. In this paper, we will define a roadmap for a rod robot in three dimensions. This roadmap is the rod hierarchical generalized Voronoi graph (rod-HGVG). Recall from Canny's roadmap work [1] that for each connected component of free space, a roadmap has the following properties: (i) accessibility, (ii) departability, and (iii) connectivity. This means that if there exists a path between two configurations, we can determine a path by first finding a path to the roadmap from the start configuration, then following the roadmap, and then finding another path from the roadmap to the destination. If a planner can incrementally construct a roadmap, then it has in essence explored the configuration space.

Canny's original roadmap work required an \textit{a priori} information about the robot's configuration space, and hence was difficult to implement. The probabilistic community [5], [9] has successfully demonstrated the capabilities of probabilistic roadmaps for highly articulated robots; their approach does not construct the configuration space, but requires knowledge of the work space prior to the planning event. Rimon [8] first suggested a method for constructing roadmap using "critical point" sensors, but did not provide details on these sensors. Choset and Burdick developed a method to construct the Voronoi Graph in the plane and in three dimensions without requiring \textit{a priori} workspace or configuration space information [2]. Later, Choset extended this result to a rod-shaped robot operating in the plane [3]. This method enabled a rod robot to map its configuration space without any prior information of its workspace (and configuration space). Loosey speaking, the rod-HGVG in $\mathbb{R}^3$ can be viewed as a combination of two prior roadmaps: (i) the point-HGVG, which is a roadmap for the point robot operating in a three-dimensional space, and (ii) the planar Rod-HGVG, which is a roadmap for rod robot operating in the plane. The rod-HGVG will be defined in terms of the distance information that sensors can easily provide, and it can be generated in an unknown environment.

1.1 The Generalized Voronoi Graph in $\mathbb{R}^2$ and $\mathbb{R}^3$

Ó'Dúnlaing and Yap [7] first applied the \textit{generalized Voronoi Diagram} (GVD) to path planning of a disk-shaped robot. This result requires full knowledge of the environment to construct the roadmap and is restricted to the plane. Choset and Burdick extended this result beyond the plane by defining the \textit{generalized Voronoi graph} (GVG) [2] which is a roadmap for a point robot operating either in the plane or in three dimensions. The GVG can be constructed using line of sight sensor data because it is defined in terms of the distance function, $d_i(r) = \min ||r - c||$, where $r \in FS$, $FS \subset \mathbb{R}^3$ is the free space and $C_i$ is an obstacle. A point-\textit{two-equidistant face} is defined as

$$F_{ij} = \{ r \in FS : d_i(r) = d_j(r), \nabla d_i(r) \neq \nabla d_j(r) \text{ and } d_i(r) \leq d_k(r) \forall h \neq i, j \}$$

(1)

In planar case, $F_{ij}$ is one-dimensional and the GVG is the union of $F_{ij}$'s, i.e., for an environment with $n$ obstacles, the GVG is $\bigcup_{i=1}^{n} \bigcup_{j=i+1}^{n} F_{ij}$. In $\mathbb{R}^3$, the point-\textit{three-equidistant face} defines the GVG. That is,

$$F_{ijk} = \{ r \in FS : d_i(r) = d_j(r) = d_k(r), \nabla d_i(r) \neq \nabla d_j(r), \nabla d_i(r) \neq \nabla d_k(r), \nabla d_j(r) \neq \nabla d_k(r), \text{ and } d_i(r) \leq d_k(r) \forall h \neq i, j, k \}$$

(2)
A three-equidistant face can also be defined as the intersection of two-equidistant faces, i.e., $F_{ijk} = F_{ij} \cap F_{jk} \cap F_{ik}$. However, we assume that the intersections are transversal [2] which is a different way of saying that the obstacles all lie in general position. The GVG is the union of the $F_{ijk}$'s, is one-dimensional, and not necessarily connected in $\mathbb{R}^3$. To connect the GVG, we will need to define additional structures, resulting in the hierarchical generalized Voronoi graph (HGVG) [2]. To distinguish the GVG from the rod-GVG, in later sections, we call this GVG defined in this section as point-GVG.

1.2 The Rod-HGVG in $SE(2)$

O'Dunlaing, Sharir and Yap [6] extended their disk-result to rod-shaped robots, but this result also requires full knowledge of the environment and is restricted to the plane. Cox and Yap developed an “on-line” strategy for rod path-planning [4], but this result does not provide a roadmap of the rod robot’s free space. In this section, we describe the rod-hierarchical generalized Voronoi graph (rod-HGVG) in the plane. This structure provides the groundwork for the rod-robot roadmap in three dimensions.

The rod in the plane has three degrees-of-freedom, and we represent a rod configuration as $q = (x, y, \theta)^T$. As with the point-GVG, we first define the distance function. Let $q$ be a rod configuration, and $R(q)$ be the set of all points that the rod occupies at configuration $q$. Then, the distance between the obstacle $C_i$ and the rod at $q$ is defined as

$$D_i(q) = \min_{r \in R(q), c \in C_i} ||r - c||.$$ 

The rod-GVG is the set of configurations equidistant to three obstacles. More formally, we first define the rod-two-equidistant face as follows

$$CF_{ij} = \{ q \in SE(2) : D_i(q) = D_j(q) \leq D_k(q) \forall k \neq i, j \text{ and } \nabla D_i(q) \neq \nabla D_j(q) \}.$$ 

Then, the rod-three-equidistance face is defined as

$$CF_{ijk} = CF_{ij} \cap CF_{jk} \cap CF_{ik}.$$ 

In the planar case, we term the rod-three-equidistant faces as rod-GVG edges. The rod-GVG is simply the union of the rod-GVG edges, i.e., the set of rod configurations that are three-way equidistant. The rod-four-equidistance faces, which can be defined similarly, are termed rod-meet points, which correspond to configurations of the rod where rod-GVG edges intersect and terminate.

The rod-GVG edges are not necessarily connected, even in the planar case. To produce a connected structure we introduce another type of edge, called R-edges.

An R-edge is the set of rod configurations defined as,

$$R_{ij} = \{ q \in \{ c(RF_{ij}) : r \in F_{ij} \} \text{ and (i) } 0 \leq d_i(r) \leq d_i(r_1) \forall r_1 \in R(q) \text{ and (ii) } d_i(r) \leq D_k(q) \forall k \neq i, j \},$$

where $RF_{ij}$ is the collection of lines tangent to $F_{ij}$ (which can be viewed as a “tangent bundle” of $F_{ij}$) [3]. Roughly speaking, R-edges connect the disconnected rod-GVG edges using point-GVG edges. The rod-HGVG then comprises Rod-GVG edges and R-edges. Figures 1 and 2 show an example of the rod-HGVG in $\mathbb{R}^2$.

We demonstrate connectivity of the rod [8] by defining a piece-wise retraction $H : FS \times [0, 1] \rightarrow CF_{ijk}$. This $H$ function also describes how the rod accesses the rod-GVG. The rod accesses the planar rod-GVG (and hence the rod-HGVG) via two gradient ascent operations: while maintaining a fixed orientation, the rod moves away from its closest obstacle and then while maintaining double equidistance and a fixed orientation, the rod moves away from the two closest obstacles until it reaches triple equidistance. Note that $\theta(q) = \theta(H(q, t))$ for all $t \in [0, 1]$, i.e., the rod arrives to the rod-GVG with a fixed orientation.

In order to “make” $H$ continuous, we divide the configuration space $R^2 \times S^1$ into junction regions $J_{ijk}$ where each junction region $J_{ijk}$ is the pre-image of $CF_{ijk}$ under $H$. This essentially guarantees that $CF_{ijk}$ is a retract of $J_{ijk}$. Finally, note that if the rod is “small” enough, $CF_{ijk}$ has one connected component and is homeomorphic to $S^1$.

For each connected component of a junction region $J_{ijk}$, there is a connected component of $CF_{ijk}$. If motion planning should occur only in one junction region, then planning is trivial because $CF_{ijk}$ is a retract of $J_{ijk}$. If the union of the $CF_{ijk}$'s formed a connected set, then planning is trivial again. In general, the $CF_{ijk}$'s will not form a connected set in $\mathbb{R}^2 \times S^1$, so we use the point-GVG to connect the $CF_{ijk}$'s of the junction regions.
The connections between junction regions are the rod configurations tangent to the point GVG, i.e., the R-edges.

Essentially, we are forming an exact cellular decomposition of $R^2 \times S^1$ where the junction regions are the cells. Planning in a cell is achieved with the rod-GVG edges, $CF_{ijk}$, and planning between the cells is achieved with the R-edges. We take an analogous approach to defining the rod-HGVG for the rod in three dimensions.

2 Rod-HGVG in $R^3 \times S^2$, based on the point-GVG

In this section, we define the rod-HGVG. Since the rod’s configuration space $R^3 \times S^2$ has five dimensions, it is natural to first define a five-way equidistant structure which we term rod-GVG edges. Just like the planar rod-GVG, the three-dimensional rod-GVG (henceforth called the rod-GVG), is not necessarily connected. Just like the planar case, we decompose $R^3 \times S^2$ into cells, also called junction regions, and “connect” them with 1-tangent edges, structures that are analogous to R-edges. However, the rod-GVG edges by themselves are not retracts of the junction regions. Instead, the set of configurations equidistant to four obstacles forms a retract of a junction region. This four-way equidistant structure has two dimensions, and thus this structure, with the 1-tangent edges and rod-GVG edges do not form a roadmap. Therefore, we define an additional structure called 2-tangent edges, which are four-way equidistant with an additional constraint. The rod-HGVG comprises rod-GVG edges, 1-tangent edges, and 2-tangent edges. This section formally defines these structures and the next two sections establish that the rod-HGVG is a roadmap.

2.1 Rod-GVG Edges

The rod-GVG is the set of configurations that are equidistant to five obstacles. The definition of the distance function in $R^3 \times S^2$ is identical to that in SE(2), i.e., $D(q) = \min_{r \in R(q), \theta \in \mathbb{C}} |r - c|$, but the rod configuration $q$ is parametrized by $q = (x, y, z, \theta, \varphi)^T$. Note that $\theta(q)$ and $\varphi(q)$ define the orientation of the rod. The rod gradient is derived in Appendix A. With distance and its gradient defined, we can define rod-equidistant faces, continuing from where we left off in the planar case. The rod-four-equidistant face is

$$CF_{ijkl} = CF_{ijkm} \cap CF_{iklj} \cap CF_{ijlk}.$$  

Then the rod-five-equidistant face, which is a rod-GVG edge in the three-dimensional case, is defined as

$$CF_{ijkmnr} = CF_{ijkl} \cap CF_{iklm} \cap CF_{jkm}.$$  

A rod-meet point is then $CF_{ijkmnr}$, the zero-dimensional set of rod configurations that are six-way equidistant.

Fig. 3. Here, there are two set of rod configurations that are five-way equidistant. The configurations in $CF_{12345}$ are equidistant to $C_1, C_2, C_3, C_4$, and $C_5$, and the configurations in $CF_{12346}$ are equidistant to $C_1, C_2, C_3, C_4$ and $C_6$. Note that they are not connected to each other.

Fig. 4. Side view of Fig. 3 with $L > W, L > H$, but here, the length of the rod is smaller than $W - H$, thus, the rod cannot be four-way equidistant to $C_2, C_3, C_4$ and $C_5$. Therefore, five-way equidistant configurations do not exist either.

The rod-GVG edges can be generated by tracing the roots of the following function

$$G(q) = \left( \frac{(D_i - D_j)(q)}{(D_i - D_k)(q)} \right).$$

In the rod configuration space $R^3 \times S^2$, the collection of rod-GVG edges is not necessarily connected, and in many environments, it may not even exist at all. Figure 3 shows a rectangular environment, where $L > W > H$ and $CF_{12345}$ and $CF_{12346}$ are the rod-GVG edges. If $L$ is substantially larger than the length of the rod, there cannot be any six-way equidistant configurations, so these two component cannot be connected. Now, if $W$ is also substantially larger than the length of the rod, the five-way equidistant configuration cannot exist (Figure 4).

2.2 1-tangent edges

The rod-GVG defined above will not necessarily be connected. Just like the rod-GVG in the planar case, we use the point-GVG to define additional structures that connect disconnected rod-GVG edge. These structures are similar to the $R$-edges where the rod is “tangent” to the point-GVG edge. More specifically, when $C_i, C_j$ and $C_k$ define a point-GVG edge $F_{ijk}$, we presume that

- The rod is “tangent” to $F_{ijk}$ at $r$.
- At $r$, $d_n(r) < d_n(r_o)$ for all other points $r_o$ on the rod, and for all three closest obstacles $C_n, n = i,j,k$, i.e., the point $r$ is the closest to all three ob-
stales.
The second condition asserts that \( r_i = r_j = r_k \), where \( r_i, r_j \) and \( r_k \) are the closest points on the rod to each obstacles [3].

Instead of calling this an R-edge, we term this structure the 1-tangent edge because it is tangent to a one-dimensional structure from the point-GVG. We can use the same technique as in the planar case to construct R-edge to construct this 1-tangent edge. The rod finishes the tracing a 1-tangent edge when it (i) reaches a four-way equidistant configuration, (ii) reaches an obstacle boundary, or (iii) detects the cycle. A cycle in the GVG is a disconnected edge diffeomorphic to S^1. Note that we consider only environments where the point-GVG is connected, and thus the condition (iii) will not occur.

2.3 2-tangent edges

Consider two 1-tangent edges that terminate at configurations equidistant to the same four obstacles. These two end point configurations cannot coincide with each other because we assume that the point-equidistant faces transversally intersect each other. In other words, two 1-tangent edges cannot intersect each other. The set of four-way equidistant configurations is two-dimensional and thus we need an additional constraint to define a one-dimensional structure that connects the two 1-tangent edges.

Consider the rod-four-equidistant face \( CF_{ijkl} \) with 1-tangent edge \( R_{ijk} \) and \( R_{ijl} \) each terminating at configurations in \( CF_{ijkl} \) (boundedness of the workspace assures this can happen). For \( R_{ijkl} \), the rod lies in the tangent space associated with \( F_{ijkl} \); restated, the rod simultaneously lies in the tangent spaces of \( F_{ij}, F_{ik}, \) and \( F_{jk} \). Likewise, for \( R_{ijl} \), the rod lies in tangent spaces \( F_{ij}, F_{ik}, \) and \( F_{jl} \). Note that in both cases the rod lies in the tangent space associated with \( F_{ij} \). Therefore, as our additional constraint, to travel from \( R_{ijkl} \) to \( R_{ijl} \) along \( CF_{ijkl} \), the rod must remain in the tangent space of \( F_{ij} \). The edge formed on \( CF_{ijkl} \) with the additional constraint of staying in the tangent space of \( F_{ij} \) is termed a 2-tangent-edge because the rod lies in a two-dimensional tangent space (as opposed to a one-dimensional tangent space with the 1-tangent edge). Note that for \( CF_{ijkl} \), there are four 1-tangent edges that terminate on the set, and there can be six different 1-tangent edges on it.

As an example, the Figure 5 shows the case where \( F_{ij} \) is a plane. Here, the rod is “tangent” to \( F_{floor} \) when the rod is moving from \( q_2 \) to \( q_3 \), and “tangent” to \( F_{floor} \) when moving from \( q_1 \) to \( q_2 \). Generally, when the rod is tracing four equidistant configuration from a configuration tangent to \( F_{ijkl} \) to a configuration tangent to \( F_{ijkl} \), it must be in tangent space of \( F_{ijkl} \).

Fig. 5. In figure (a), \( q_1, q_2, q_3 \) and \( q_4 \) represent four-way equidistant configurations. Also they are terminal configurations of different 1-tangent edges. Figures (b), (c) and (d) shows 2-tangent edges that connect some pairs of 1-tangent edges.

Figures (a), (b), (c) and (d) represent four-way equidistant configurations. Also they are terminal configurations of different 1-tangent edges. Figures (b), (c) and (d) show 2-tangent edges that connect some pairs of 1-tangent edges.

First we need to define some additional structures. We define the set of rod configurations \( q \) in \( CF_{ij} \) that satisfy the conditions (i) \( q \) is tangent to \( F_{ij} \), (ii) \( r_i = r_j \), where \( r_i \) and \( r_j \) are the points on the rod closest to each obstacles \( C_i \) and \( C_j \) as \( RC_{ij} \), i.e.,

\[
RC_{ij} = \{ q \in cl(RF_{ij}) : r \in F_{ij} \text{ and } \begin{align*} 
  (i) &\quad 0 \leq d_i(r) \leq d_i(r_j) \forall r_j \in R(q) \text{ and } \nonumber \\
  (ii) &\quad d_i(r) \leq D_i(q) \forall h \neq i,j \}. 
\end{align*}
\]

where, \( RF_{ij} \) is the set of planes tangent to \( F_{ij} \) (which is analogous to a tangent bundle of \( F_{ij} \)). We form the 2-tangent edge \( R_{ijkl} \) by intersecting \( RC_{ij} \) and \( CF_{ijkl} \), i.e., \( R_{ijkl} = RC_{ij} \cap CF_{ijkl} \).

Note that \( RC_{ij} \cap CF_{ijkl} = RC_{ij} \cap CF_{ijkl} \). As shown in Appendix B, \( RC_{ij} \) is three dimensional. Since \( CF_{ijkl} \) and \( RC_{ij} \) are each three dimensional, \( R_{ijkl} \) is generally one-dimensional by the pre-image theorem.

The \( R_{ijkl} \) can be constructed by tracing the roots of

\[
G(q) = \begin{pmatrix} 
(D_i - D_j)(q) \\
(D_i - D_j)(q) \\
(D_i - D_j)(q) \\
(D_i - D_j)(q) \\
(r_i - r_j)(q) \\
\end{pmatrix}
\]

Here, the first three elements ensures the four-way equidistance. The fourth element forces the two closest points on the rod to \( C_i \) and \( C_j \) to be the same and thus specifies the orientation of the rod. Since sensors can easily provide distance information and determine the closest points on the rod to the closest obstacles, the 2-tangent edge can be readily constructed without a priori information.
3 Accessibility

The accessibility of the rod-GVG (and hence the rod-HGVG) via four gradient ascent operations: the first three use a fixed orientation gradient \( \nabla D_i(x) \) which directs the rod to increase distance to an object \( C_i \) while maintaining its orientation; the last gradient ascent operation uses the full gradient \( \nabla D_i(x) \) which is derived in the Appendix. The first three gradient ascent operations implicitly define a function \( H^3: FS \times [0,1] \rightarrow CF_{ijk} \) that is analogous to the \( H \) mapping which enables a rod to access the planar rod-GVG. First, while maintaining a fixed orientation, the rod moves away from its closest obstacle (i.e., \( \hat{c}(t) = \nabla D_i(c(t)) \)), then while maintaining double equidistance and a fixed orientation, the rod moves away from the two closest obstacles until it reaches triple equidistance (i.e., \( \hat{c}(t) = \pi_{\tau_{x_i}(CF_{ij})} \nabla D_i(c(t)) \)), where \( \pi_{\tau_{x_i}(CF_{ij})} \) is the projection operator, and then while maintaining triple equidistance and a fixed orientation, the rod moves away from the three closest obstacles until it reaches a four-way equidistance (i.e., \( \hat{c}(t) = \pi_{\tau_{x_i}(CF_{ijk})} \nabla D_i(c(t)) \)).

Finally, we define another mapping \( HJ: CF_{ijk} \rightarrow R_{ij/k} \cup CF_{ijkm} \) that moves the rod away from the four closest obstacles, using the full gradient,

\[
\hat{c}(t) = \pi_{\tau_{x_i}(CF_{ijk})} \nabla D_i(c(t))
\]

This step terminates at either of the following: (i) five-way equidistance, i.e., a rod-GVG edge, or (ii) a 2-tangent edge (proof appears in Appendix C). Once the rod has accessed the rod-HGVG, it can then begin incrementally constructing the rod-HGVG using the previously defined numerical techniques. If the rod-HGVG is connected, then numerically constructing it ensures complete exploration of a connected component of the rod’s configuration space.

4 Connectivity

We will demonstrate that if the point-GVG is connected, then there exists a path between two rod configurations \( q_1, q_2 \in \mathbb{R}^2 \times S^2 \) if and only if there exists a path between two rod configurations \( q_1, q_2 \) on the rod-HGVG in \( \mathbb{R}^2 \times S^2 \). Note that \( q_1, q_2 \) are the “projected” configurations onto the rod-HGVG from \( q_1, q_2 \) via a mapping \( HJ \circ H^3 : \mathbb{R}^2 \times S^2 \times [0,1] \rightarrow \) rod-HGVG.

The \( H^3 \) mapping is analogous to a retraction and is defined in a similar fashion as \( H \) in the planar case. In \( \mathbb{R}^2 \times S^2 \), we will take a similar approach as the planar case to “make” \( H^3 \) continuous; here, we define junction regions that are \( J_{ijk} \) and the connections among them are the 1-tangent edges \( R_{ijk} \), the set of rod configurations tangential to the point-GVG edge \( CF_{ijk} \).

A junction region \( J_{ijk} \) is the pre-image of \( CF_{ijk} \) under \( H^3: \mathbb{R}^2 \times S^2 \times [0,1] \rightarrow CF_{ijk} \). Again, \( H^3 \) is implicitly defined by a sequence of three fixed-orientation gradient ascent operations. Note that \( \theta(q) = \theta(H^3(q,1)) \) and \( \phi(q) = \phi(H^3(q,1)) \) and using a similar approach as in [3], \( H^3 \) can be shown to be continuous in each junction region \( J_{ijk} \). Therefore, \( CF_{ijk} \) is a two-dimensional retract of \( J_{ijk} \). In fact, if the rod is “small” enough, \( CF_{ijk} \) is diffeomorphic to a two-sphere, \( S^2 \). At this point, we can infer that if the point-GVG is connected in \( \mathbb{R}^2 \), then there exists a path between \( q_1 \) and \( q_2 \) in \( \mathbb{R}^2 \times S^2 \) if and only if there exists a path between \( H^3(q_1,1) \) and \( H^3(q_2,1) \) in the union of 1-tangent edges \( R_{ijk} \) and rod-four-equidistant faces \( CF_{ijk} \).

We have not defined a roadmap yet because \( CF_{ijk} \) has two dimensions, not one. To define a one-dimensional structure on \( CF_{ijk} \), we define another mapping \( HJ: CF_{ijk} \rightarrow R_{ij/k} \cup CF_{ijkm} \). This function is also implicitly defined via a gradient ascent operation. From Section 3, we showed that once the rod achieves four-way equidistance, continuous gradient ascent (full gradient ascent) brings the rod either to a two-tangent edge \( R_{ij/k} \) or a rod-GVG edge \( CF_{ijk} \).

If the rod is “small” enough, then this last gradient ascent operation brings the rod only to a two-tangent edge. If the rod is “large” enough, the two-spheres \( CF_{ijk} \) and \( CF_{ijkm} \) intersect and form an edge \( CF_{ijkm} \) and gradient ascent could bring the rod to a five-way equidistant configuration (but can also bring the rod to the rod-four-equidistant face). If the rod is even larger (or obstacles are appropriately shaped), a rod-meet-point occurs when three two-sphere intersect.

For the sake of discussion, let’s assume the rod is “small” enough so no two-sphere intersect with each other. In other words, let’s assume there are no rod-GVG edges and rod-meet-points on the rod-four-equidistant faces. If \( HJ \) were continuous on a rod-four-equidistant face, then the two-tangent edges would form a retract of the rod-four-equidistant faces and we are done.

Alas, this is not the case. Instead, we define another cellular decomposition on \( CF_{ijk} \) where each cell is denoted \( J_{ij/k} \) and is defined by the pre-image of \( R_{ij/k} \) under the \( HJ \) mapping. For each cell \( J_{ij/k} \), \( HJ \) is continuous and therefore \( R_{ij/k} \) is a one-dimensional retract of \( J_{ij/k} \).

The remaining challenge is to establish that the two-tangent edges “link up” properly. This is easily shown because each end point of the 1-tangent edges coincides with the end point of three different 2-tangent edges. Consider the 1-tangent edge \( R_{ijk} \); it has two types of end points: a boundary configuration and a four-way equidistant configuration (i.e., equidistant to \( C_i \) in addition to \( C_j \) and \( C_k \)). Consider the end point that is four-way equidistant. By definition of the 1-tangent edge, its end point is tangent to the point-GVG edge \( F_{ijk} \), which means that the end point is also tangent to
the point-two-equidistant faces $F_{ij}$, $F_{jk}$ and $F_{kh}$. Therefore, this end point also belongs to the 2-tangent edges $R_{ij/kh}$, $R_{ik/jh}$ and $R_{ik/hj}$. This shows that three of the six 2-tangent edges connect. By repeating this argument for the other combination of 1-tangent edges and 2-tangent edges, one can easily see how all of the 2-tangent edges link up.

5 Simulation Result

We performed a computer simulation using the algorithm described in this paper. Figures 6 and 7 shows the complete rod-HGVG in simple environments. For more simulation results, see http://voronoi.sbp.ri.cmu.edu.

6 Conclusion

This paper introduces a roadmap called rod hierarchical generalized Voronoi graph for a rod-shaped robot operating in a three-dimensional space. The rod-HGVG is defined in terms of workspace distance information, which sensors can easily provide. Using workspace distance information, we can prescribe a sensor-based incremental method to achieve motion planning without constructing the configuration space. This is important for sensor based planning because we cannot construct configuration space without knowing the workspace first. Moreover, even when we have configuration space representation, it is still difficult to measure the distance in configuration space than in workspace. The rod-HGVG has three components: (i) rod-GVG edges, which are five-way equidistant, (ii) 2-tangent edges, which are four-way equidistant and (iii) 1-tangent edges, which are three-way equidistant.

The first contribution of this paper is the piece-wise retract: we defined an exact cellular decomposition of the rod’s configuration space where in each cell, called a junction region, we defined a retract that makes motion planning in the junction region trivial. We then use the point-GVG to connect the retracts of each junction region, which is the second contribution of this paper. We were able to use connectivity of a structure defined in the robot’s workspace to infer connectivity properties of the robot’s configuration space. In other words, when connectivity of the point-GVG represents the connectivity of the works space, then connectivity of the rod-HGVG represents the connectivity of the rod’s configuration space.

In general the point-GVG is not connected in R^2, so future work will use the point-HGVG to connect the junction regions. We started with the point-GVG only to achieve a tractable subgoal. We also defined a piece-wise retract on $CT_{ij}$ to guarantee connectivity of the rod-HGVG. This relied on $HJ$ being continuous, which due to space limitations, was not detailed in this paper; a rigorous proof should be demonstrated.

Ideally, we would like to demonstrate this on a robot blimp. This work is a step towards the ultimate goal of sensor based planning for an articulated multi-body robot. The next step is to consider two rod robots, and then an n-rod robot.

References

[9] Steven A. Wilmarth, Nancy M. Annato, and Peter F. Stiller. Motion planning for a rigid body using random networks on
Appendix

A Rod Gradient

Here, we derive the rod distance gradient. In Fig. 8, $X$, $Y$, and $Z$ denote the world coordinate frame, and $X_r$, $Y_r$, and $Z_r$ denote the body fixed coordinate frame on $R$ (the rod). Let $(x, y, z)^T$ be the origin of the body fixed coordinate frame, and $\theta, \varphi$ be the orientation of the body fixed coordinate frame with respect to the world coordinate frame. Let $c$ be the closest point on the obstacle $C_i$ to the robot and $r$ be the closest point to $C_i$ on the robot. Finally let $(a, b, c)^T$ be the coordinate of $r$ in body fixed coordinate frame. Then the world coordinate of $r$ is

$$r = \begin{bmatrix} x + a \cos \theta \cos \varphi - b \sin \theta + c \cos \theta \sin \varphi \\ y + a \sin \theta \cos \varphi + b \cos \theta + c \sin \theta \sin \varphi \\ z - a \sin \varphi + c \cos \varphi \end{bmatrix}$$

Then, following similar steps to the two dimensional case [3], the first three components of the distance gradient are

$$\frac{\partial D}{\partial x} = \frac{1}{D_i(q)} \left[ (r_x - c_x), (r_y - c_y), (r_z - c_z) \right]$$

The remaining two rotational components are

$$\frac{\partial D}{\partial \theta} = \frac{1}{D_i(q)} \left[ a \cos \theta \varphi + b \theta + c \cos \theta \sin \varphi, -a \sin \theta \varphi + b \cos \theta + c \sin \theta \sin \varphi \right]$$

$$\frac{\partial D}{\partial \varphi} = \frac{1}{D_i(q)} \left[ a \cos \theta \varphi + b \theta + c \cos \theta \sin \varphi, a \varphi \sin \theta + b \varphi \cos \theta + c \varphi \sin \theta \sin \varphi \right]$$

B Dimension of $RC_{ij}$

In this section, we show that $RC_{ij}$ is three-dimensional. Note that for any rod configuration $q \in CF_{ij}$, $R(q) \cap F_{ij}$ is not empty, and without loss of generality, we assume that $R(q) \cap F_{ij}$ is zero dimensional, i.e., a single point. For rod configurations in $RC_{ij}$, $r_i = r_j = R(q) \cap F_{ij}$. We call that point the “contact point” and simply denote it as $r$. We define the constant distance curve $cv_{ij}(\Omega)$ on $F_{ij}$ as the set of all points on $F_{ij}$ that have distance $\Omega$ to $C_i$, i.e.,

$$cv_{ij}(\Omega) = \{ x \in F_{ij} : d_i(x) = \Omega \}.$$ 

Now we consider the contact condition and dimensionality of $RC_{ij}$, in two separate cases, depending on the contact point $r$ for each rod configurations: (I) $d_i(r)$ is at local minimum of the distance on $F_{ij}$ to $C_i$ on $F_{ij}$, (II) otherwise.

Lemma B.1 For rod configurations in $CF_{ij}$ that are tangent to $F_{ij}$ and $r_i = r_j$, such that $d_i(r)$ is not a local minimum of the distance (i.e., Case II above), (a) the contact point $r$ must be either an end point $P$ or $Q$, or (b) the rod must be in the tangent space of $cv_{ij}(d_i(r))$ and the contact point can be an interior point of the rod. Also, when the rod satisfies case (II) above, the rod has one degree of freedom in this set when the contact point is fixed.

Proof: Note that, in a neighborhood about the origin of $T_rcv_{ij}, T_rcv_{ij}(d_i(r))$ separates the $T_rF_{ij}$ into two regions, such that all the points in one region have distance less than $d_i(r)$ to $C_i$, and the points in the other region have greater distance. For case (a), by hypothesis, the rod lies in a line that transversally intersects $T_rcv_{ij}(d_i(r))$. So, if $r \neq P, Q$, the rod itself intersects $T_rcv_{ij}(d_i(r))$ in its interior. So loosely speaking, $P$ and $Q$ are on opposite sides of $T_rcv_{ij}(d_i(r))$.

Let’s define a set $L_i(\Omega)$, such that

$$L_i(\Omega) = \{ x \in \mathbb{R}^3 : d_i(x) = \Omega \}.$$ 

This set is a two dimensional manifold in $\mathbb{R}^3$ (Note that $cv_{ij}(d_i(r))$ can be seen as the intersection of $L_i(d_i(r))$ and $F_{ij}$. Because the rod intersects $T_rcv_{ij}(d_i(r))$ at $r$ and $r$ is on $L_i(d_i(r))$, the rod must “poke” into $L_i(d_i(r))$. This contradicts the condition that the point $r$ is the closest point on the rod to each obstacle. So the rod cannot intersect the $cv_{ij}(d_i(r))$ in its interior, i.e., the intersection must occur at a rod’s end point $P$ or $Q$, when the intersection in transversal.

If the rod “touches” $cv_{ij}$ at $P$ or $Q$ (case (a)), the rod cannot translate, but can rotate around the contact point as long as it "stays" on the tangent space $T_rF_{ij}$. When the rod is "tangent" to $cv_{ij}$ (case (b)), the rod cannot change its rotation, but translate along its length. Therefore, when the contact point is fixed on $F_{ij}$, the rod has one degree of freedom in each case. Because $F_{ij}$ is two dimensional, the rod has three degrees of freedom in $RC_{ij}$. Figure 9 describe this case. ■

Lemma B.2 For case (i) above, the contact point can be one of the interior points on the rod, and the rod has two degrees of freedom at each configuration in this set.
Recall that when the rod configuration belongs to $R_{ij} - R_{ij}^1$, the rod must be "tangent" to $c_{ij}(d)$ for some value of $d$. Now if the rod moves outside the set $R_{ij}^1$, the $d_{ij}$ distances to $C_i$ and $C_j$ increase. Let's denote that the new distance as $d_i(r) + \varepsilon$. Then, from the continuity of the distance function the $c_{ij}(d_i(r) + \varepsilon)$ is a closed curve around the point $r$. The rod must be "tangent" to this curve, and there are only two points on that curve where the tangent is parallel to the rod. That means that the rod has (only) one more additional degree of freedom (which moves the rod outside of the $R_{ij}^1$), thus makes the rod lost contact at $r$, but still the rod stays in $R_{ij}$ at the given configuration in $R_{ij}^1$. Thus the rod has three degrees of freedom in this case. (Figure 10) □

C Accessibility Proof

In this section, we demonstrate that a rod starting at a four-way equidistant configuration will access either (i) a rod-GVG edge or (ii) a 2-tangent edge via full gradient ascent of distance to the four closest obstacles.

Proving case (i) is trivial. Now, we prove case (ii), i.e., the gradient ascent will eventually direct the rod onto the $R_{ij/kil}$ when there are no five-way equidistant configurations. Figure 11(a) shows a rod configuration which intersects $SS_{ik}$ nontraversally. Consider the plane formed by the rod vector $PQ$ and the vector $r_k C_k$. Then let the global coordinate frame be located such that (i) the origin is located on $P$, (ii) the X and Y axes are on the plane formed by $PQ$ and $r_k C_k$. Then the Z axis of the global coordinate frame is perpendicular to that plane. In this frame, the rod configuration is at $(0,0,0,0,0)$ and the body fixed coordinate of the $r$ is $(a,0,0)$ with some positive value $a$. Note that in this coordinate system, the $\theta$ component of the gradient rotates the rod in the plane, and the $\phi$ component of the gradient rotates the rod out of the plane. We interested in the $\theta$ component, which for the obstacle $C_k$ is

$$\frac{\partial D_k}{\partial \theta} = \frac{1}{\sqrt{c_k}} \left[c_k - r_k^2, c_k^2 - r_k^2, 0^T, (a_k \sin \theta, -a_k \cos \theta, 0) \right]^T.$$

Then, the two vectors $\langle c_k^2 - r_k^2, a_k \cos \theta, 0 \rangle$ and $\langle a_k \sin \theta, -a_k \cos \theta, 0 \rangle$ are parallel. So $\frac{\partial D_k}{\partial \theta}$ has positive value, and that means that the angle between the tangent plane on $SS_{ik}$ and the rod increases as a result of gradient ascent. Now do the same analysis with $r_k C_l$; the vector $r_k C_l$ is in the "opposite" direction of $\langle a_k \sin \theta, -a_k \cos \theta, 0 \rangle$ and the coordinate value of $a_l$ is larger because $r_l$ is on the other side of $SS_{ik}$. This means that the gradient for $C_l$ has a negative component of $\frac{\partial D_k}{\partial \theta}$ on plane defined by $PQ$ and $r_k C_l$, with larger value than $\frac{\partial D_k}{\partial \theta}$, i.e., $\frac{\partial D_k}{\partial \theta} < \frac{\partial D_k}{\partial \theta}$. This decreases the angle between the tangent plane and the rod. Thus, the angle between the tangent plane and the rod will decrease during the gradient ascent, forcing the rod to become tangent.
Fig. 11. The rod is two-way equidistant to obstacles $C_k$ and $C_l$.

The rod intersects $SS_k,l$ transversally. In second figure, the rod is shown on the plane defined by $PQ$ and $r_k,e_k$. Also in this figure $r_l Q_l$ is on the same plane, but in general it will be on different plane.