MODEL REDUCTION OF FLEXIBLE MANIPULATORS

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Abstract

Flexible manipulators can be characterized by a dynamic model with a large number of vibration modes, and the use of the model in the model-based control schemes requires reduction of model order. Balanced truncation is an effective method for model reduction of asymptotically stable systems by transforming the states to a coordinate system in which the controllability and observability Gramians are equal and diagonal, and eliminating the states which contribute weakly to the input-output map. An elastic flexible manipulator, however, is a *marginally stable system* and thus the balanced truncation method can not be directly applied. In this report, we present a method of reducing the order of a marginally stable system based on the fact that translation transformations in the frequency domain preserve input-output properties of the system. We address the successful application of the method to model reduction of flexible manipulators with infinite-dimensional or finite-dimensional model. The method is also applicable for any other marginally stable model, such as elastic space trusswork and multi-dimensional space vehicle structure.

1 Introduction

Flexible manipulators have been increasingly interesting to many researches for light-weight, energy-efficiency, and less harmfulness concerns. Modeling and controlling such manipulators, however, is a challenge because conceptually the system must be characterized by a distributed parameter systems (DPS) with infinite number of modes, while controlling DPS requires a finite-dimensional model. A various approaches of methods can be employed to reduce the order of a model with either an infinite-dimensional or a finite-dimensional structure. For example, model order can be reduced by retaining the first finite number of modes and truncating the rest. In this case, the criterion for selecting appropriate number of modes is not clearly known and truncation usually results in a model with higher or lower order than that is needed. Another method frequently used is assuming the order of a model based on lumping mass model such as Lagrangian approach. Again, this method also has difficulty in selecting appropriate mode shapes to generate an accurate model with a minimal number of modes.

The balanced truncation method [1] is based on the realization theory that for zero initial condition, the input-output behavior of the model is completely specified by the mutually controllable and observable subspace R^n. The procedure is to transform state space description of a stable system to balanced coordinates such that the input-to-state and state-to-output couplings are weighted equally. This implies that observability and controllability Gramians are equal and diagonal. The diagonal elements of the balanced Gramians are in fact a set of closed-loop input-output invariants, or the Hankel singular values, which characterize the contribution of each state to the input-output map of the system. The states that are weakly controlled and weakly observed have "small" singular values and the states that contribute weakly to input-output map can be deleted.

Model reduction by balancing techniques is applicable only for stable systems [1]–[5]. This is because, for the balancing techniques, the computation of the controllability and observability Gramians requires that the system be asymptotically stable. Some researches [6]–[9] have been directed to applying balancing method to flexible structure by assuming lightly damped and widely separated modes. This assumption is normally not valid in practice. For a flexible manipulator, as an elastic system, we have to face the fact that the system is marginally stable. There are very few research articles discussing the model reduction of a marginally stable or unstable system. In the paper [10], it has been shown that an unstable non-minimal realization (A, B, C) can be transformed via a similarity transformation, into a balanced one, if and only if the product of the controllability and observability Gramians is similar to a real diagonal matrix Λ.

This report presents a method to reduce the model order of flexible manipulators using balancing techniques. The approach is based on the fact that translation transformations in the frequency domain preserve input-output properties of the system. In section 2, we describe the modeling of two flexible manipulators with an infinite or finite dimensional structure. In section 3, we present the theory of balancing model reduction technique for a marginally stable system where the poles are on the imaginary axis. In section 4, we address the application of the method to model reduction of flexible manipulators.

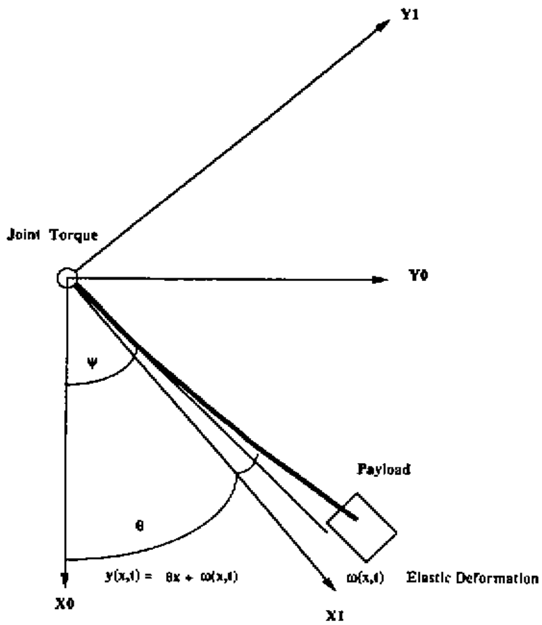


Figure 1: The One Link Flexible manipulator

2 Modelling of Flexible Manipulators

In this section, we will discuss the modeling of flexible manipulators with infinite-dimensional and finite-dimensional structures. These two models will be used to demonstrate the procedure of model reduction in section 4.

2.1 Infinite-dimensional model

The infinite-dimensional model for a one-link flexible manipulator shown in Figure 1 can be modeled [11] under the following assumptions : (1) motion is planar and payload and joints are symmetric objects in XY plane. (2) rotary inertia and shear deformation effects are neglected. (3) gravitation force is in the Z direction. (4) beam inertia and flexibility are uniformly distributed over the link length.

Based on these assumptions, the dynamics of the position of any point along the link is governed by the Bernoulli-Euler beam equation:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho a \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{1}$$

where $0 < x < l$ and the equation has associated boundary conditions.

The transfer function for general Bernoulli-Euler beam equation can be found in [12] and the complete dynamics can be described by

$$\frac{\partial^2 Q(x,t)}{\partial t^2} + a^2 \frac{\partial^4 Q(x,t)}{\partial x^4} = f(x,t) \tag{2}$$

$$Q(x,0) = Q_0(x), \quad \frac{\partial Q}{\partial t}(x,0) = Q_1(x), \quad \frac{\partial^2 Q}{\partial x^2}(0,t) = g_1(t)$$
$$\frac{\partial^3 Q}{\partial x^3}(0,t) = g_2(t), \quad \frac{\partial^2 Q}{\partial x^2}(l,t) = g_3(t), \quad \frac{\partial^3 Q}{\partial x^3}(l,t) = g_4(t)$$
$$0 \le x \ge l, \quad t \ge 0, \quad a > 0$$

To solve the above problem, we may define a *standardizing function*:

$$\begin{aligned} w(x,t) &= f(x,t) + Q_0(x)\delta'(t) + Q_1\delta(t) - a^2\delta'(x)g_1(t) + a^2\delta(x)g_2(t) - \\ &\quad -a^2\delta(l-x)g_3(t) - a^2\delta(l-x)g_4(t) \end{aligned} \tag{3}$$

and we convert the original problem into a zero initial and boundary condition problem for which the Green's function is

$$G(x,\xi,t) = \frac{4}{a}\sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\mu_n^2 \varphi_n^2(l)} \sin a\mu_n^2 t \tag{4}$$

where

$$\begin{aligned} \varphi_n(x) &= (\sinh \mu_n l - \sin \mu_n l)(\cosh \mu_n x + \cos \mu_n x) - \\ &\quad -(\cosh \mu_n l - \cos \mu_n l)(\sinh \mu_n x + \sin \mu_n x) \end{aligned}$$

and the μ_n are the non-negative roots of the equation $\cosh \mu l \cos \mu l = 1$.
The transfer function is given by:

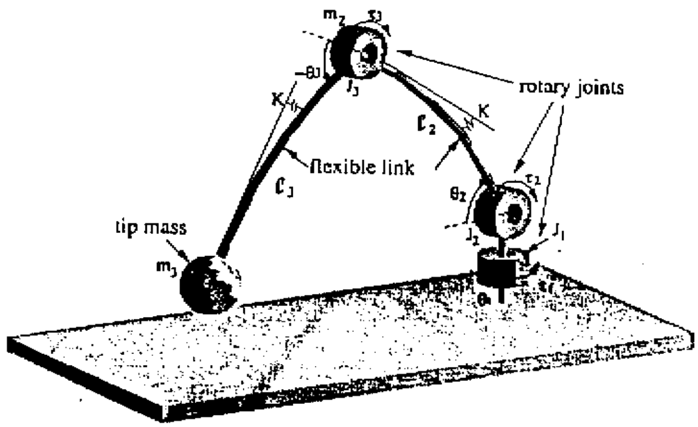
$$W(x,\xi,s) = \frac{4}{a}\sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\varphi_n^2(l)} \frac{1}{s^2 + a^2\mu_n^4} \tag{5}$$

which has infinite poles:

$$s = \pm ja\mu_n^2, \qquad n = 1, 2, \ldots$$

2.2 Finite-dimensional model

For a multi-degree of freedom flexible manipulator, Lagrangian dynamics is an appropriate approach to model the system. As an example, the Self-Mobile Space Manipulator developed in the Robotics Institute of Carnegie Mellon University [13][14] can be modeled as shown in Figure 2. Two flexible links are connected by the flex joint and the end of the link holds the



Parameters

l_i	$i = 1-3$	length of link i
m_i	$i = 2, 3$	mass of joint i
J_i	$i = 1-3$	joint inertia i with high gear ratio. ($J_i = n_i^2 J_{mi}$) n_i : gear ratio J_{mi} : motor inertia
K_{bi}	$i = 2, 3$	bending stiffness of link i, ($K_{bi} = \frac{E_i I_{bi}}{l_i^3}$) E_i : Young's modulus I_{bi} : moment of cross section area inertia
K_{ti}	$i = 2, 3$	torsional stiffness on link i, ($K_{ti} = \frac{G_i I_{ti}}{l_i}$) G_i : Shear modulus I_{ti} : polar moment of area inertia

Variables

θi	$i = 1-3$	angular displacement of joint i
v_{yi}, v_{zi}	$i = 2, 3$	bending deflection at the end of link i
ϕ_{xi}	$i = 2, 3$	torsional deflection of link i
ϕ_{yi}, ϕ_{zi}	$i = 2, 3$	angular bending deflection of link i
τ_i	$i = 1-3$	torque on joint i

Figure 2: The Two Link Flexible Manipulator

tip mass, while the other end of the link is connected to the flex and twist joints attached to the base. Each compact joint contains a driving motor and a high reduction gearing, and each flexible link is slender tubing to maximize its strength with minimum weight.

The following assumptions are made to simplify the model:(1) The middle joint and the end-effector are lumped as a point mass and the moment of inertia of the motor in the joint is considered for the high reduction gearing. (2) The mass and the moment of inertia of the links are neglected, compared to the joints and tip mass, or the moment of inertia of the motor. (3) Each link possesses high tensile and shear stiffness so that no tensile and share distortion are assumed. Compliance in the bending and torsional directions of the link is considered. (4) The deflections of the link is small with respect to the length of the link. (5) The gravity effect is not taken into account for space applications.

The kinetic energy is a summation of the rotational kinetic energy of three revolute joints and the translational kinetic energy of two point mass. The potential energy consists of torsional strain energy and bending strain energy. Based on the kinetic energy and potential energy, Lagrangian model is as follows.

$$T = \sum_{j=1}^{3} \frac{1}{2} J_j \dot{\theta}_j^2 + \sum_{j=2}^{3} \frac{1}{2} m_j (\dot{x}_{m_j}^2 + \dot{y}_{m_j}^2 + \dot{z}_{m_j}^2) \tag{6}$$

$$\begin{aligned} U &= \sum_{j=2}^{3} \frac{1}{2} K_{t_j} \phi_{x_j}^2 + \sum_{j=2}^{3} K_{b_j} (3v_{y_j}^2 - 3\ell_j v_{y_j} \phi_{z_j} + \ell_j^2 \phi_{z_j}^2) \\ &\quad + \sum_{j=2}^{3} K_{b_j} (3v_{z_j}^2 - 3\ell_j v_{z_j} \phi_{y_j} + \ell^2 \phi_{y_j}^2) \end{aligned} \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = f_i \tag{8}$$

where

$$\begin{aligned} \boldsymbol{q} &= \left[\, q_1 \cdots q_{13} \,\right]^T \\ &= \left[\, \theta_1\ \theta_2\ v_{y_2}\ v_{z_2}\ \phi_{x_2}\ \phi_{y_2}\ \phi_{z_2}\ \theta_3\ v_{y_3}\ v_{z_3}\ \phi_{x_3}\ \phi_{y_3}\ \phi_{z_3} \,\right]^T \\ \boldsymbol{f} &= \left[\, f_1 \cdots f_{13} \,\right]^T \\ &= \left[\, \tau_1\ \tau_2\ 0\ \cdots\ \cdots\ \cdots\ 0\ \tau_3\ 0\ \cdots\ \cdots\ \cdots\ 0 \,\right]^T \end{aligned}$$

The joint angular position and link deformation in torsional and bending directions are selected as a set of generalized displacements, and the joint torques and zero elements form a set of generalized forces. By a standard procedure based on Lagrangian dynamics, the explicit dynamics equation can be obtained.

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{K}(\boldsymbol{q})\boldsymbol{q} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{f} \tag{9}$$

where M is the inertia matrix, K is the stiffness matrix, and C is the centrifugal and Coriolis torques.

The dynamic equation above is *nonlinear due to the translational kinetic energy part* in Equation (7). To linearize the model with reasonable accuracy, we assume that the speed of the robot is low so that centrifugal and Coriolis torques are negligible and the effect of link deformation on the variation of the inertia matrix is small so that the inertia matrix is a function of only joint angles. In this way the dynamic model can be linearized.

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f} \tag{10}$$

It is obvious that the poles of the model lie on the $j\omega$ axis.

The model can be decoupled into two independent parts, tangential model and radial model. The tangential model represents the motion associated with joint 1 driving the tip mass in the circular direction, while the radial model describes the motion associated with joints 2 and 3 driving the tip mass in the plane perpendicular to the circular direction. It must be noted that the mass and stiffness matrices for both models are functions of the mixed joint angles. In this way, we can obtain two linearized models in the tangential and radial directions

$$\bar{\boldsymbol{M}}_t\bar{\boldsymbol{q}}_t'' + \bar{\boldsymbol{K}}_t\bar{\boldsymbol{q}}_t = \bar{\boldsymbol{\tau}}_t \tag{11}$$

$$\bar{\boldsymbol{M}}_t = \begin{bmatrix} j_1 + m\ell^2\cos^2\theta_2 + \ell_x^2 & m\ell\cos\theta_2 + \ell_x & \ell_x \\ m\ell\cos\theta_2 + \ell_x & m+1 & 1 \\ \ell_x & 1 & 1 \end{bmatrix}$$

$$\bar{\boldsymbol{K}}_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3k\left(\frac{\cos^2\theta_3}{k\ell^2} + \frac{\sin^2\theta_3}{k_t} + \frac{1}{3}\right)}{\Delta_1} & \frac{-\frac{6\cos\theta_3}{4\ell}}{\Delta_1} \\ 0 & \frac{-\frac{6\cos\theta_3}{4\ell}}{\Delta_1} & \frac{1}{\Delta_1} \end{bmatrix}$$

$$\begin{aligned} \bar{\boldsymbol{q}}_t &= \begin{bmatrix} \bar{\theta}_1 & \bar{v}_{x_2} & \bar{v}_{x_3} \end{bmatrix}^T \\ &\equiv \begin{bmatrix} \theta_1 & \frac{v_{x_2}}{\ell_3} & \frac{1}{\ell_3}(v_{x_3} + \ell_3\sin\theta_3\phi_{x_2} + \ell_3\cos\theta_3\phi_{y_2}) \end{bmatrix}^T \\ \bar{\boldsymbol{\tau}}_t &= \begin{bmatrix} \bar{\tau}_1 & 0 & 0 \end{bmatrix}^T \\ &\equiv \begin{bmatrix} \frac{\tau_1}{K_{b_3}\ell_3^2} & 0 & 0 \end{bmatrix}^T \end{aligned}$$

$$j_1 \equiv \frac{J_1}{m_3 \ell_3^2}, \qquad m \equiv \frac{m_2}{m_3}, \quad \ell \equiv \frac{\ell_2}{\ell_3}$$

$$\ell_x = \frac{\ell_2}{\ell_3}\cos\theta_2 + \cos(\theta_2+\theta_3) \quad k \equiv \frac{K_{b_2}}{K_{b_3}}, \quad k_t \equiv \frac{K_{t_y}}{K_{b_3}\ell_3^2}$$

$$\Delta_1 \equiv \frac{\cos^2\theta_3}{4k\ell} + \frac{\sin^2\theta_3}{k_t} + \frac{1}{3}, \qquad \omega \equiv \sqrt{\frac{K_{b_3}}{m_3}} \quad q' \equiv \frac{dq}{d\omega}$$

$$\bar{M}_r \bar{q}_r'' + \bar{K}_r \bar{q}_r = \bar{\tau}_r \tag{12}$$

$$\bar{M}_r = \begin{bmatrix} j_2 + m\ell^2 + \ell^2 + 1 + 2\ell\cos\theta_3 & m\ell + \ell + \cos\theta_3 & 1 + \ell\cos\theta_3 & 1 + \ell\cos\theta_3 \\ m\ell + \ell + \cos\theta_3 & m+1 & \cos\theta_3 & \cos\theta_3 \\ 1 + \ell\cos\theta_3 & \cos\theta_3 & 1 + j_3 & 1 \\ 1 + \ell\cos\theta_3 & \cos\theta_3 & 1 & 1 \end{bmatrix}$$

$$\bar{K}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{12k\ell(k\ell+3)}{\Delta_2} & 0 & -\frac{18k\ell}{\Delta_2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{18k\ell}{\Delta_2} & 0 & \frac{12k\ell^2}{\Delta_2} \end{bmatrix}$$

$$\bar{q}_r = \begin{bmatrix} \bar{\theta}_2 & \bar{v}_{y_2} & \bar{\theta}_3 & \bar{v}_{y_3} \end{bmatrix}^T \equiv \begin{bmatrix} \theta_2 & \frac{v_2}{\ell_3} & \theta_3 & \frac{v_3 + \ell_3\phi_{z_2}}{\ell_3} \end{bmatrix}^T$$

$$\bar{\tau}_r = \begin{bmatrix} \bar{\tau}_2 & 0 & \bar{\tau}_3 & 0 \end{bmatrix}^T \equiv \begin{bmatrix} \frac{\tau_2}{K_{b_3}\ell_3^2} & 0 & \frac{\tau_3}{K_{b_3}\ell_3^2} & 0 \end{bmatrix}^T$$

$$j_2 \equiv \frac{J_2}{m_3\ell_3^2}, \qquad j_3 \equiv \frac{J_3}{m_3\ell_3^2},$$

$$\Delta_2 \equiv 4k\ell^2 + 3$$

3 Model Reduction for a Marginally Stable System

From the discussion above, it has been known that the model of flexible manipulator is marginally stable, i.e., the poles of the system lie on imaginary axis. In this section, we present a method of model reduction for such a system.

Consider the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{13}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input or the control vector, $y(t) \in R^p$ is the output vector, and A, B, C are matrices of appropriate sizes.

The transfer function of the system (1) is given by

$$H(s) = C(sI - A)^{-1}B \tag{14}$$

Definition 1: The symbol (A, B, C) represents a state space realization of the transfer function of the system.

Definition 2: The realization (A, B, C) is said to be minimal if it is completely controllable and observable.

Definition 3: (A, B, C) is said to be asymptotically stable if Re $(\lambda \mid A \mid) < 0$.

Definition 4: Let (A, B, C) be a minimal and asymptotically stable realization. (A, B, C) is said to be balanced if A, B and C satisfy the matrix Lyapunov equations with equal and diagonal controllability and observability Gramians, that is

$$AW_c + W_cA^T = -BB^T \tag{15}$$
$$A^TW_o + W_oA = -C^TC, \tag{16}$$

where $W_c = W_o = \Sigma = diag[\sigma_1\ \sigma_2\ \cdots \sigma_n]$ and Hankel singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq 0$.

The matrix Σ can be partitioned into two submatrices Σ_1 and Σ_2 in the following way

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{17}$$

where $\Sigma_1 = diag[\sigma_1\ \sigma_2\ \cdots \sigma_k]$ and $\Sigma_2 = diag[\sigma_{k+1}\ \sigma_{k+2}\ \cdots \sigma_n]$.

Further partition A, B, C corresponding to the partition of Σ

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1\ \ C_2], \tag{18}$$

Then the kth order reduced order model is (A_{11}, B_1, C_1).

Definition 5: A minimal realization (A, B, C) is said to be a J form realization if A is a Jordan matrix J and (J, B, C) has following structure:

$$J = diag[J_1\ J_2\ \cdots J_\alpha], \quad B = [B_{i1}\ B_{i2} \cdots B_{i\alpha}],$$
$$C = [C_{i1}\ C_{i2} \cdots C_{i\alpha}] \tag{19}$$

with

$$J_{ij} = diag[J_{ij_1}\ J_{ij_2}\ \cdots J_{ij_{r_{ij}}}], \quad B_{ij} = [B_{ij_1}\ B_{ij_2}\ \cdots B_{ij_{r_{ij}}}],$$
$$C_{ij} = [C_{ij_1}\ C_{ij_2} \ldots C_{ij_{r_{ij}}}]$$

and

$$J_{ijk} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda_i & 1 & 0 \\ \vdots & \vdots & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix}$$

$$J_{ij1} = J_{ij2} = \cdots = J_{ijr_{ij}}, \quad J_{ijk} \in C^{\alpha_{ij} \times \alpha_{ij}}$$
$$B_{ijk} = [b_{ijk_1} b_{ijk_2} \cdots b_{ijk_{\alpha ij}}], \quad B_{ijk} \in C^{m \times \alpha_{ij}}$$
$$C_{ijk} = [c_{ijk_1} c_{ijk_2} \cdots c_{ijk_{\alpha ij}}], \quad C_{ijk} \in C^{m \times \alpha_{ij}}.$$

If $(A, \tilde{B}, \tilde{C})$ is a minimal realization, there exists a non-singular matrix T such that

$$J = T^{-1}AT, \;\; B = T^{-1}\tilde{B}, \;\; C = \tilde{C}T \tag{20}$$

$$H(s) = C(sI - J)^{-1}B \tag{21}$$

In the rest of this section, we present an approach to model reduction of marginally stable system using balancing technique. We at first state the following theorems.

Theorem 1: The controllability and observability of a minimal realization J-form (J, B, C) depends only on the structures of J, B and C.

The proof of Theorem 1 can be obtained directly from the definitions of the controllability and observability.

Theorem 2: For minimal marginally stable realization $H(s)$, there exist translation transformations in frequency domain, $s' = s + a$, such that the poles of $H(s + a)$ lie strictly on the left half s-plane and the controllability and observability of $H(s+a)$ are identical to those of $H(s)$.

Proof: If $H(s)$ is a minimal realization, we have

$$H(s) = C(sI - J)^{-1}B$$

and

$$H(t) = L^{-1}[H(s)]$$

Let λ_{max} be the largest eigenvalue of the matrix J. For the marginally stable system $Re(\lambda_{max}) = 0$. we can choose

$$a = \varepsilon$$

where $\varepsilon > 0$.
Let $\hat{H}(t) = H(t)e^{-at}$, then the Laplace transform of $\hat{H}(t)$ can be determined by

$$\hat{H}(s) = H(s+a) = C[(s+a)I - J]^{-1}B = C(sI - \hat{J})^{-1}B$$

Thus

$$Re(\lambda \mid \hat{J} \mid) < 0$$

Since rank(J) = rank$(\hat{J})$, $(\hat{J}, B, C)$ is a minimal realization and J and $\hat{J}$ have same structure. From Theorem 1, $(\hat{J}, B, C)$ and (J, B, C) have the same controllable and observable subspace.

In the balanced realization, the *ith* Hankel singular value characterizes how controllable and observable the *ith* state is. Therefore the model can be reduced by neglecting the states corresponding to "small" Hankel singular values. This implies that the criterion for balanced model reduction depends only on the input-output information of the system. Thus it is possible to obtain a reduced order model by balancing method for an asymptotically stable system if the input-output map of the system can be determined. More specifically, we transform the marginally stable system into the asymptotically stable system by the translation transformation in the frequency domain and balance the resulting stable system. This procedure leads to the following algorithm.

Algorithm

1. Transform $H(s)$ into $\hat{H}(s) = H(s+a)$ so that $\hat{H}(s)$ has no poles on the right half s-plane nor on the $j\omega$ axis.

2. Find a minimal state space realization (A, B, C) of $\hat{H}(s)$.

3. Compute the balanced realization (A_b, B_b, C_b) from (A, B, C). That is

   $$A_b = T^{-1}AT, \quad B_b = T^{-1}B, \quad C_b = CT,$$

   where T is the transformation matrix which makes the controllability and observability Gramians equal and diagonal.

   T can be obtained using any of methods presented in [1]–[5].

4. Determine the reduced order model (A_r, B_r, C_r) from (A_b, B_b, C_b). Recall that for the balanced realization, $W_c = W_o = \Sigma = diag[\sigma_1\ \sigma_2\ \cdots \sigma_n]$ and Hankel singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq 0$. If $\sigma_k \gg \sigma_{k+1}$, then Σ can be partitioned as

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where $\Sigma_1 = diag[\sigma_1\ \sigma_2\ \cdots \sigma_k]$ and $\Sigma_2 = diag[\sigma_{k+1}\ \sigma_{k+2}\ \cdots \sigma_n]$.

(A_b, B_b, C_b) can be partitioned accordingly

$$A_b = \begin{bmatrix} A_{b_{11}} & A_{b_{12}} \\ A_{b_{21}} & A_{b_{22}} \end{bmatrix}, \quad B_b = \begin{bmatrix} B_{b_1} \\ B_{b_2} \end{bmatrix}, \quad C_b = [C_{b_1}\ \ C_{b_2}]$$

to yield the reduced order model

$$A_r = A_{b_{11}}, \quad B_r = B_{b_1}, \quad C_r = C_{b_1}.$$

5. Obtain $\hat{H}_r(s)$ from (A_r, B_r, C_r) as

$$\hat{H}_r(s) = C_r(sI - A_r)^{-1}B_r.$$

6. The reduced order model of the original system is given by

$$H_r(s) = \hat{H}(s - a).$$

Some remarks may be in order:

1. The translation transformation, $s' = s + a$, is a conformal transformation which preserves congruence and involves neither rotation nor stretching. Therefore the translation transformation in the frequency domain does not affect the input-output description of the model.

2. The algorithm can be interpreted in the time domain as follows

$$H(t) = H(t) \cdot e^{-at} \cdot e^{at} = \hat{H}(t) \cdot e^{at} \tag{22}$$

$$H_r(t) = H_r(t) \cdot e^{-at} \cdot e^{at} = \hat{H}_r(t) \cdot e^{at}, \tag{23}$$

where $H(t)$ and $\hat{H}(t)$ are the impulse responses of the full order model and the modified full order model, $H_r(t)$ and $\hat{H}_r(t)$ are the impulse responses of the reduced order model and the modified reduced order model, respectively. Equation (23) suggests that the impulse response of an unstable system $H(t)$ can be factored into a stable impulse $\hat{H}(t)$ multiplied by e^{at}, while Equation (24) implies that $H_r(t)$ is equal to $\hat{H}_r(t)$ multiplied by e^{at}. Therefore, if

$$\hat{H}_r(t) \approx \hat{H}(t)$$

then

$$H_r(t) \approx H(t).$$

3. Some model reduction methods, such as Routh approximation, may be applicable to the unstable model reduction, but they may result in a stable reduced order model and make implementation of model-based control scheme difficult. For the proposed method, the poles which lie on the imaginary axis will be the dominant poles of the modified system and thus balancing will keep properties of these poles.

4. The method can be applied equally well to both SISO and MIMO systems. In MIMO case, $H(s)$ is a transfer function matrix.

4 Model Reduction of Flexible manipulators

As discussed in section 2, the flexible manipulator can be modeled as either infinite-dimensional model or finite-dimensional system. In either case, it is generally needed to reduce the order of the model so that any model-based control scheme can be implemented in practice. In this section, we discuss the application of the method proposed in the previous section to the two models developed in section 2.

Example 1: Infinite-dimensional model

We list five modes of the infinite-dimensional model based on the parameters suggested in [11] in Table 1.

Table 1: Poles and zeros of the flexible manipulator

Pair	Poles	Zeros
1	0.00	±11.18
2	±15.42j	±60.45
3	±49.97j	±149.27
4	±104.25j	±277.58
5	±178.27j	±445.35

This is a tenth order model and is difficult to be implemented in real-time control. Using our method, we can reduce the model order and the resultant second order model is

$$H_r(s) = \frac{s^2 + 1.6013 \times 10^4 s - 7.6801 \times 10^5}{s^2 + 3.3671 \times 10^{-3} s - 3.3552 \times 10^{-4}}$$

The step responses of the tenth order model and the corresponding second order model are compared in Figure 3. It has been shown that the second order model is nearly identical to the tenth order model.

Example 2: Finite-dimensional model

The second example of flexible manipulators is the radial direction model developed in section 2 which has two inputs and four outputs and is manipulator configuration dependent. At $\theta_2 = \theta_3 = 60$ degree, the transfer function is given by

$$NUM_{11} = s^2(2.6823e+01s^4 + 3.7052e+03s^2 + 9.1806e+03)$$

$$NUM_{12} = s^2(1.7131e+03s^2 - 1.3589e+04)$$

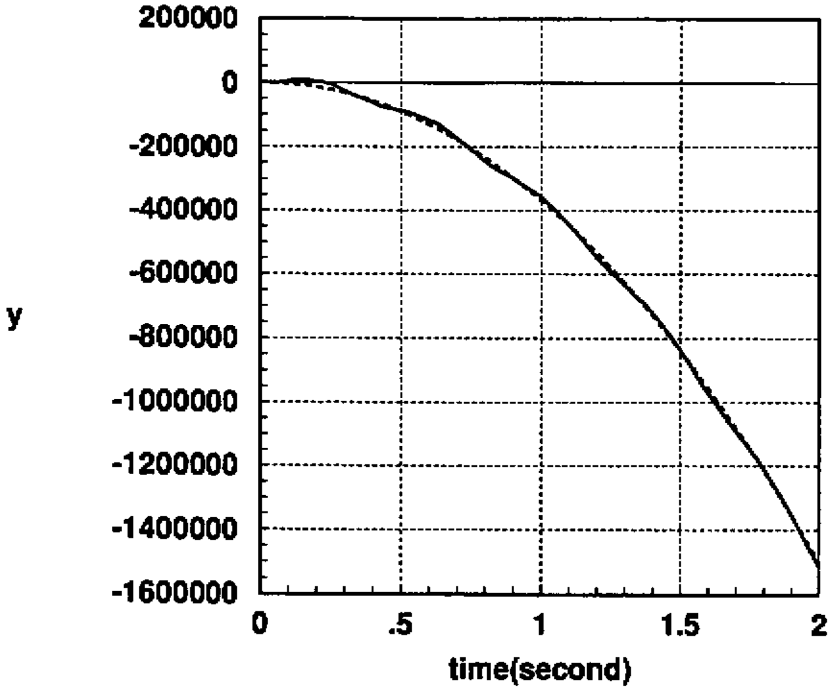
$$NUM_{21} = s^4(-2.6823e+01s^2 - 3.5937e+03)$$

$$NUM_{22} = -1.9382e+03s^4$$

$$NUM_{31} = s^2(1.7131e+03s^2 - 1.3589e+04)$$

$$NUM_{32} = s^2(7.4510e+01s^4 + 7.6274e+03s^2 + 3.0999e+04)$$

$$NUM_{41} = s^4(-2.6823e+01s^2 - 5.4970e+03)$$



— **Original model ...Reduced order model**

Figure 3: Step Responses of the Full and Reduced Order Models

$NUM_{42} = s^4(-7.4510e + 01s^2 - 9.1002e + 03)$

$DEN = s^4(s^4 + 2.3010e + 02s^2 + 1.1032e + 04)$
where NUM_{ij}/DEN represents the transfer function from the torque τ_i to the angle θ_j. This is a 8th order MIMO system and using our method the reduced to 7th order is as follows.

$NR_{11} = 3.4686e - 01s^6 + 2.6394e + 01s^5 + 7.6040e + 01s^4 + 3.6210e + 03s^3 + 3.3186e + 03s^2 + 7.0192e + 03s - 9.9127e + 02$

$NR_{12} = 1.6638e - 01s^6 - 3.4956e - 01s^5 + 3.8582e + 01s^4 + 1.6517e + 03s^3 + 1.6239e + 03s^2 - 1.4633e + 04s + 2.0666e + 03)$

$NR_{21} = s^2(-4.6450e - 04s^4 - 2.6773e + 01s^3 + 3.7313s^2 - 3.5851D + 03s + 5.0513e + 02)$

$NR_{22} = s^2(-2.2280e - 04s^4 + 2.4323e - 02s^3 - 6.9621e - 02s^2 - 1.9340e + 03s + 2.7257e + 02)$

$NR_{31} = 1.6667e - 01s^6 - 2.5671e - 01s^5 + 3.8477e + 01s^4 + 1.6660e + 03s^3 + 1.6091e + 03s^2 - 1.4628e + 04s2.0660e + 03)$

$NR_{32} = 7.9945e - 02s^6 + 7.4317e + 01s^5 + 7.9596s^4 + 7.5946e + 03s^3 - 1.7942e + 02s^2 + 3.0496e + 04s - 4.3073e + 03)$

$NR_{41} = s^2(-4.1424e - 04s^4 - 2.6699e + 01s^3 + 3.6426s^2 - 5.4776e + 03s + 7.7092e + 02)$

$NR_{42} = s^2(-1.9869e - 04s^4 - 7.4450e + 01s^3 + 1.0504e + 01s^2 - 9.0908e + 03s + 1.2826e + 03)$

$DENR = s^2(s^5 - 1.4289e - 01s^4 + 2.3010e + 02s^3 - 3.2765e + 01s^2 + 1.1032e + 04s - 1.5582e + 03)$
where $NR_{ij}/DENR$ represents the transfer function of the reduced order model from the torque τ_i to the angle θ_j. The step responses of the original model and the reduced order model are compared in Figure 4.

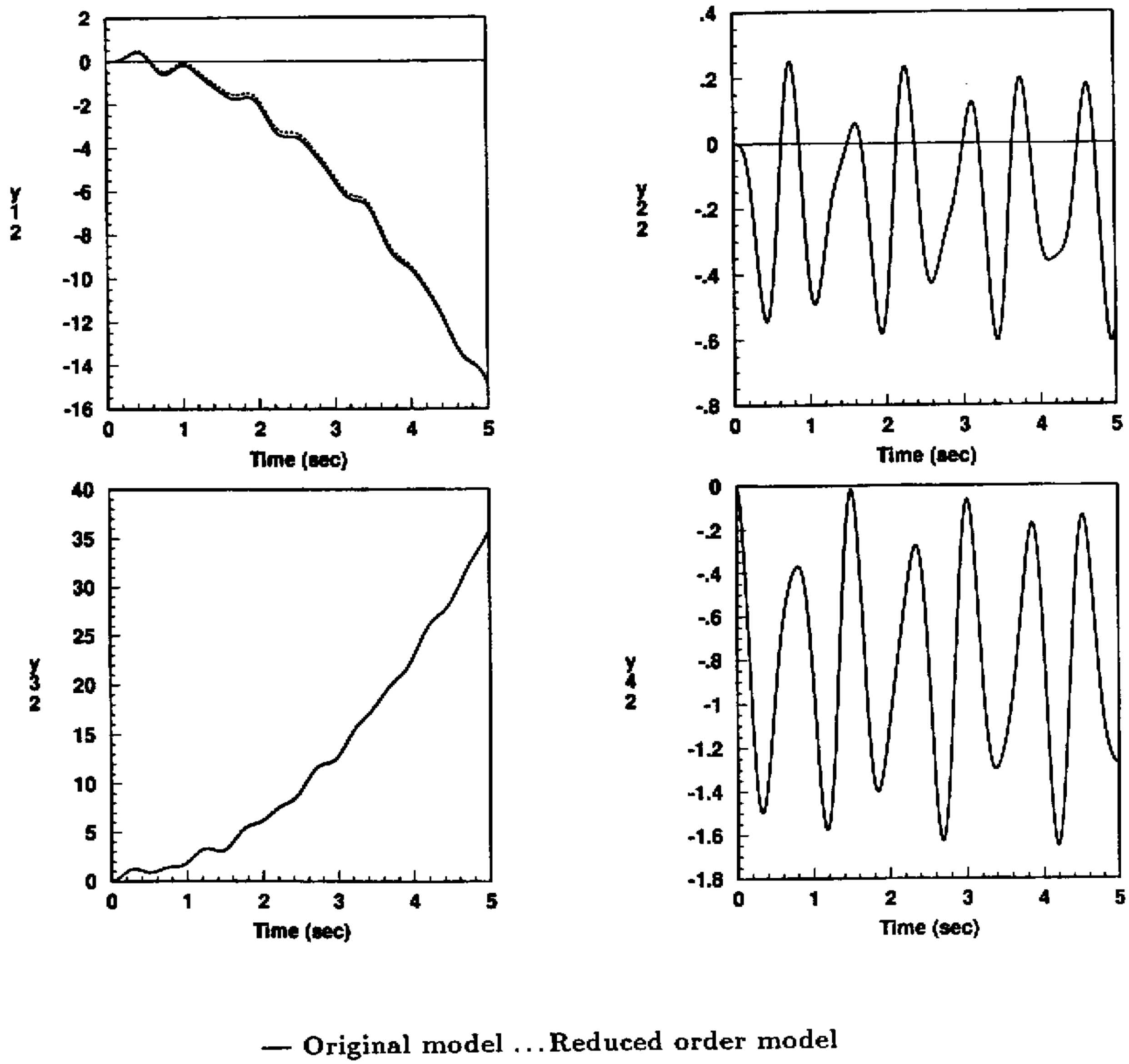


Figure 4: Step Responses of the Full and Reduced Order Models

5 Summaries

We presented a simple, efficient method for model reduction of flexible manipulators based on the fact that the translation transformation in s-plane preserves input-output property of the system. By translation transformation in frequency domain, we can change the stability of the system without lose of its input-output property. The original balancing method requires that the model be asymptotically stable while this method is applicable to a marginally stable system. We employed the proposed method to reduce the order of flexible manipulator models such that model-based scheme can be implemented. The method has been examined by two examples, an infinite-dimensional model and a finite-dimensional model.

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