Skewed Symmetry Groups*

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Abstract

We introduce the term skewed symmetry groups and provide a complete theoretical treatment for 2D wallpaper groups under affine transformations. For the first time, a given periodic pattern can be classified not simply by its Euclidean symmetry group but by its highest "potential" symmetry group under affine deformation. A concise wallpaper group migration map is constructed that separates the 17 affinely deformed wallpaper groups into small, distinct orbits. The practical value of this result includes a novel indexing and retrieval scheme for regular patterns, and a maximal-symmetry-based method for estimating shape and orientation from texture under unknown views.

1 Motivation

When studying periodic patterns, a useful fact from mathematics is the answer to Hilbert's 18th problem: there is only a finite number of symmetry groups for all possible periodic patterns in dimension n [1]. There are seven *frieze groups* [23] for 2D patterns repeated along one dimension, seventeen *wallpaper groups* [21] for patterns extended by two linearly independent translational generators, and 230 space groups (regular crystals) [9, 6] extended by three linearly independent translations. The mathematics of these *Crystallographic groups* or discrete groups [5] has been used extensively in physics, chemistry, and the arts [8, 22, 28]. However, the application of these classic results to periodic pattern analysis in computer vision has yet to be fully explored.

Symmetry groups are composed of rigid Euclidean transformations. We define *skewed symmetry groups* as affinely transformed Euclidean symmetry groups. Previous results have been published on Euclidean treatment of crystallographic groups and their applications in pattern recognition [15]. Our work differs from previous work in three major aspects: (1) the space of transformations is extended to affine; (2) the relationship among the symmetry groups under affine deformation is studied thoroughly (there is no existing work on this topic even in the mathematics literature [4]); and (3) we explore novel applications such as deformable regular pattern classification by "potential symmetry", and develop a maximal-symmetry-based method for estimating shape from texture under unknown viewpoints.

The method we develop in this paper gives a constructive algorithm to compute the potential symmetry for a given planar periodic pattern under the mathematical framework of crystallographic groups [5]. One interesting finding is that given 17 distinct symmetry groups for wallpaper patterns, the number of different symmetry groups a pattern can affinely "migrate" to is very small, no more than 4. *Even though the appearance of a periodic pattern can change drastically and infinitely under affine transformations, its symmetry group stays finite and relatively stable, and thus provides a good index for regular textures viewed at arbitrary angles.* A complete and concise skewed symmetry group "migration map" is constructed that separates the intertwined relationships of the 17 affinely deformed wallpaper groups into small, distinct orbits. We have implemented a computer algorithm for skewed symmetry group classification for patterns extracted from real images.

Different from existing work on regular pattern classification, in our work a given periodic pattern is classified not simply as a rigid pattern, but as all possible affine versions of the pattern. This seemingly vast set of possibilities is uniquely constrained by the associated symmetry groups. On the one hand, two different patterns may share the same inherent skewed symmetry group, on the other hand, two patterns with the same symmetry group may not possess the same "potential symmetries" when undergoing affine transformations. Our method can discriminate these patterns. All three patterns in Figure 1 initially (leftmost patterns) have the 2-fold (180^o) rotation symmetry as their highest order of symmetry. Under all affine deformations the symmetry group of each pattern follows a different path to reach its own 'highest' order:

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(1) has 4-fold rotational symmetry, (2) has 6-fold and (3) still only has 2-fold rotation symmetry plus reflections. The advantage of using crystallographic groups

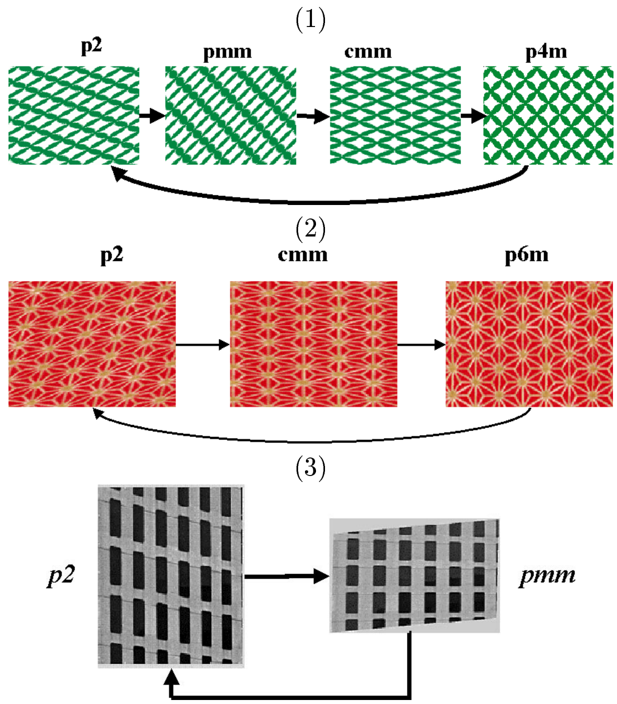


Figure 1: When a pattern is deformed by affine transformations, its symmetry group migrates to different groups within its orbit: (1) $p2 \rightarrow pmm \rightarrow cmm \rightarrow p4m$, (2) $p2 \rightarrow cmm \rightarrow p6m$, (3) $p2 \rightarrow pmm$. Note: the labels $p1, p2, p3, p6, \ldots$ are classic notations for crystallographic groups. For details see [5, 21].

as our theoretical foundation is that the deformation of patterns can be done purposefully (without heuristic search or ad hoc rules) and efficiently by a deterministic algorithm.

2 Skewed Symmetry Groups

The study of skewed symmetry groups is a conceptual extension to *skewed symmetry*[1] [11], the difference is that our study includes all possible symmetries (not limited to reflections) and takes advantage of the group theoretical nature of symmetries. Analysis of skewed symmetry has attracted several researchers [7, 18, 27, 26], where the main goal is to automatically

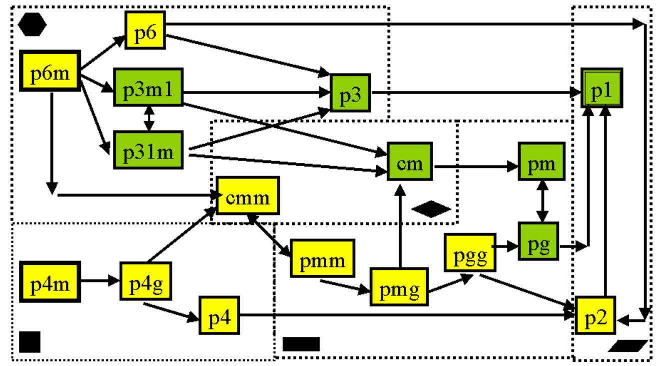


Figure 2: **Symmetry group hierarchy:** The label inside each box is the classic notation for one of the 17 wallpaper groups [5, 21]. Arrows in the diagram indicate *subgroup* relationships among the 17 wallpaper groups, where $A \rightarrow B$ means B is a subgroup of A. Note, the mutual inclusive relations $A \leftrightarrow B$ mean A contains a subgroup isomorphic to B and vice versa. The small black shapes at the corner of each dotted region indicate the basic lattice unit shape of the symmetry groups within the region.

determine the skewed reflection axis of a deformed bilateral pattern. The algorithms developed are basically local methods, seeking corresponding points on contours [7, 27] or intensity images [19] using parallelism and collinearity invariants. Rather than analyzing exact pattern features in the picture (pattern contours), our approach deals with the underlying symmetry structure of the pattern. Our algorithm examines a selected set of possible affine-invariant symmetries to determine all potential symmetry group classifications of the pattern. The "most symmetrical" group in the orbit then dictates the pattern's ultimate symmetry potential, which can be used as an affine invariant pattern index.

2.1 Affine Symmetry Invariants

A symmetry g of a periodic pattern P is a distance preserving transformation (e.g. translation, rotation, reflection, glide-reflection) that maps every pixel in the pattern to a pixel of the same appearance (e.g. grey-value or color) such that $g(P) = P$. The set of all such symmetry transformations comprise the pattern's symmetry group[2].

Figure 2 is a diagram adapted from [5] illustrating the subgroup relationship hierarchy of the 17 wallpa-

[1] A mirror reflection symmetry viewed from some (unknown) viewing direction, such that the original reflection symmetry axis is not perpendicular to the line (chords [27]) linking corresponding mirror point pairs.

[2] Mathematically speaking, the wallpaper groups are defined for 2D periodic patterns that cover the whole 2D plane. In practice, we use the concept of symmetry group G of P with the assumption that G is the symmetry group of an infinite periodic pattern for which P is a finite patch.

per groups. Figure 2 also shows different lattice unit shapes for different subsets of the 17 wallpaper groups. Note that all lattice units are parallelograms. Rectangular units have angles of 90^o. Rhombic units have equal-length edges. Square units are a special case of both rectangle and rhombic, and hexagonal units are a special case of rhombic. These five basic lattice shapes form yet another hierarchy.

In robotics manipulation related research, where only rigid transformations are considered, we have shown (Proposition 3.3.1 [16]):

Proposition: *If G is a symmetry group of P and A is a rigid transformation, then AGA^{-1} is the symmetry group of $A(P)$.*

Here, AGA^{-1} is a conjugation of group G via A [6]. When A is a rigid transformation or a uniform scaling, there exists a bijection between the original symmetry group and the conjugated symmetry group, and G and AGA^{-1} are considered equivalent (isomorphic).

When a periodic pattern undergoes non-rigid transformations (other than uniform scaling), the above proposition no longer holds. In AGA^{-1} the original symmetry group G is being skewed by A, thus the term *skewed symmetry groups*. An important thing to realize is that certain symmetries do survive some constrained or even general affine transformations. The question is whether AGA^{-1} retains its original group structure, and if not, which symmetry groups will it migrate to?

Let us examine under what conditions a symmetry of a pattern P remains when an affine transformation A is applied to P. If g is a symmetry of pattern P, $g(P) = P$ (definition of symmetry). For each $x \in P$, there exists a unique $y \in P$ such that $g(x) = y$. Let $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Applying A to P, and **assuming** g remains a symmetry of $A(P)$ so $g(A(P)) = A(P)$ (modulo translations and rotations in A acting on g since we know they preserve symmetries), we have

$$g(A(x)) = A(y) \Rightarrow g(A(x)) = A(g(x)) \Rightarrow gA = Ag \ .$$

From this relation, we derive constraints that affine transform A must satisfy to maintain symmetry g

1) When g is a 2-fold rotation (rotation by 180 degrees),

$$g = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix},$$

$$gA = \begin{vmatrix} -a & -b \\ -c & -d \end{vmatrix}, Ag = \begin{vmatrix} -a & -b \\ -c & -d \end{vmatrix}$$

There are thus no constraints on the values of a,b,c,d. This means that a 2-fold rotational symmetry is invariant to any non-singular affine transformation.

2) W.l.g. when g is a reflection about the Y axis

$$g = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix},$$

$$gA = \begin{vmatrix} -a & -b \\ c & d \end{vmatrix}, Ag = \begin{vmatrix} -a & b \\ -c & d \end{vmatrix}$$

The derived constraints are: $b = 0$ and $c = 0$. This means that a reflection is invariant **only** to nonuniform scaling parallel and perpendicular to the axis of reflection.

3) When g is an n-fold rotation where $n \neq 2$ (in this case, rotations by 120 degrees, 90 degrees and 60 degrees)

$$g = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$gA = \begin{vmatrix} a\cos\theta - c\sin\theta & b\cos\theta - d\sin\theta \\ a\sin\theta + c\cos\theta & b\sin\theta + d\cos\theta \end{vmatrix}$$

$$Ag = \begin{vmatrix} a\cos\theta + b\sin\theta & -a\sin\theta + b\cos\theta \\ c\cos\theta + d\sin\theta & -c\sin\theta + d\cos\theta \end{vmatrix}$$

Equating the two sides, we have the constraints:

$a\cos\theta - c\sin\theta = a\cos\theta + b\sin\theta \Rightarrow b = -c$

$b\cos\theta - d\sin\theta = -a\sin\theta + b\cos\theta \Rightarrow a = d$

$a\sin\theta + c\cos\theta = c\cos\theta + d\sin\theta \Rightarrow a = d$

$b\sin\theta + d\cos\theta = -c\sin\theta + d\cos\theta \Rightarrow b = -c$

Therefore

$$A = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} = \underbrace{\sqrt{a^2+b^2}}_{S} \underbrace{\begin{vmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{vmatrix}}_{R}$$

where S is a uniform scaling and R is a rotation with $\theta = \text{atan2}(b, a)$. Therefore, 3-fold, 4-fold and 6-fold rotational symmetries are only invariant to similarity transformations.

2.2 Symmetry Group Migration

Based in part on the above results, we can derive a set of conditions that specify when two **different** symmetry groups, G_1 and G_2, **can not** be transformed into each other:

1. G_1 has a 2-fold rotation symmetry but G_2 does not (2-fold rotation survives any nonsingular affine distortion);

2. G_1 and G_2 have the same lattice type;

3. after deforming the lattice type of group G_1 into the lattice type of group G_2, at least one remaining symmetry in the deformed G_1 differs from all symmetries in G_2; and

4. G_1 and G_2 do not have a subgroup relationship (Figure 2).

Table 1: **Wallpaper Group Migration Map** This is a 17x17 table where each entry indicates whether the row-group to the left of the Table can be affinely transformed into a column-group on the top of the Table by: **S** – a similarity transformation, **N** – a non-uniform scaling $\perp$ or $\|$ to all reflection axes in the group to the left, **A** – a general affine transformation other than **S** or **N**. An entry of **P** means that row-group may or may not be transformed into the column-group, depending on the particular pattern (see Figure 1 for some examples). Entries with numbers state that there are no affine deformations that can possibly transform the row-group to the column-group, and the number listed indicates which of the 4 conditions listed in Section 2.2 hold (only one number is listed when multiple justifications hold).

	p1	p2	pm	pg	cm	pmm	pmg	pgg	cmm	p4	p4m	p4g	p3	p3m1	p31m	p6	p6m
p1	**A**	1	**P**	**P**	**P**	1	1	1	1	1	1	1	**P**	**P**	**P**	1	1
p2	1	**A**	1	1	1	**P**	**P**	**P**	**P**	**P**	**P**	**P**	1	1	1	**P**	**P**
pm	**A**	1	**N**	2	3	1	1	1	1	1	1	1	4	3	3	1	1
pg	**A**	1	2	**N**	3	1	1	1	1	1	1	1	4	3	3	1	1
cm	**A**	1	3	3	**N**	1	1	1	1	1	1	1	4	**P**	**P**	1	1
pmm	1	**A**	1	1	1	**N**	2	2	**P**	3	**P**	3	1	1	1	4	3
pmg	1	**A**	1	1	1	2	**N**	2	3	3	3	3	1	1	1	4	3
pgg	1	**A**	1	1	1	2	2	**N**	**P**	3	3	**P**	1	1	1	4	3
cmm	1	**A**	1	1	1	**P**	3	**P**	**N**	3	**P**	**P**	1	1	1	3	**P**
p4	1	**A**	1	1	1	3	3	3	3	**S**	2	2	1	1	1	4	3
p4m	1	**A**	1	1	1	**N**	3	3	**N**	2	**S**	2	1	1	1	3	3
p4g	1	**A**	1	1	1	3	3	**N**	**N**	2	2	**S**	1	1	1	3	4
p3	**A**	1	4	4	4	1	1	1	1	1	1	1	**S**	2	2	1	1
p3ml	**A**	1	3	3	**N**	1	1	1	1	1	1	1	2	**S**	2	1	1
p3lm	**A**	1	3	3	**N**	1	1	1	1	1	1	1	2	2	**S**	1	1
p6	1	**A**	1	1	1	4	4	4	3	4	3	3	1	1	1	**S**	2
p6m	1	**A**	1	1	1	3	3	3	**N**	3	3	4	1	1	1	2	**S**

We can now construct a 17×17 "migration map" that lists the complete set of groups that any one of the 17 wallpaper groups can be transformed into under affine transformations (Table 1).

2.3 Skewed Symmetry Group Classification

Work presented in this paper differs from previous work on Euclidean treatment of wallpaper groups in that a given periodic pattern is no longer considered as rigid. In correspondence with their symmetry group hierarchy (Figure 2), regular patterns can now form families of hierarchies as well under all possible affine transformations. This is a very important extension since now a pattern is no longer judged by its "face value", but instead by the most symmetrical pattern it can possibly become.

We have developed an algorithm to determine all potential symmetry groups of a pattern under affine deformation. This seemingly lengthy process is made simple by using some basic facts of crystallographic groups and information from Table 1. The algorithm runs as follows

Step 1: given a periodic pattern, find the underlying translation lattice and its unit lattice using a robust peak finding algorithm (details are found in [15]) ;

Step 2: deform this unit lattice (and corresponding pattern) into a *square* lattice and classify the symmetry groups of this new pattern using a Euclidean symmetry group classification algorithm;

Step 3: repeat step 2 on a *hexagonal* shaped lattice;

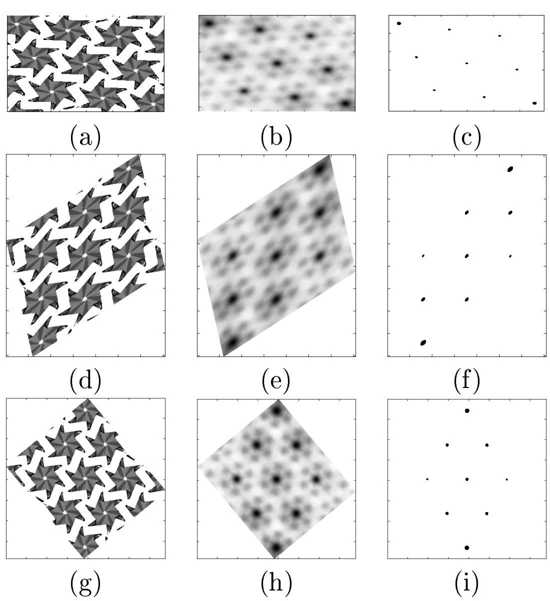
Step 4: determine which one of the two resulting groups from steps 2 and 3 is the highest symmetry group in the subgroup hierarchy (Figure 2). Use this group as the row-group index in Table 1 and find all the other potential symmetry groups that this pattern can deform into under affine transformations.

One of the key ideas in this simple algorithm is to perform an affine transformation to "normalize" the detected lattice structure into either a square or hexagonal grid. These are not arbitrary choices nor heuristics, but based on the well-defined lattice hierarchies of wallpaper groups (Figure 2). This artificially creates the best geometric condition for higher order symmetries (for example, 6-fold rotational symmetry) to appear. Meanwhile, the original symmetries of the pattern are guaranteed to be preserved under at least one of these two deformations, because hexagonal and square lattices are the most symmetric special cases of the more general lattices (Figure 2). The symmetry groups identified under these conditions are the most symmetrical ones the pattern can possibly have, even if the original pattern has a relatively low symmetry group to start with (See Figure 1). This step in the classification algorithm automatically takes care of the entries with a **P**(pattern dependent case) in Table 1.

Even though the appearance of a periodic pattern can change infinitely and drastically under affine transformations, its symmetry group stays finite and relatively stable, and thus provides a good index for regular textures viewed at arbitrary angles. A new symmetry measurement of periodic patterns can be defined as the hierarchy of all the symmetry groups that this pattern can be associated with while undergoing affine deformation. This measures one unique property of a periodic pattern: *potential symmetry.*

3 Experimental Results

The examples in this section illustrate the process of symmetry classification under unknown affine transformations. The first example uses a synthetic image from [10]. The results from Step 1 of the algorithm are shown in Figure 3: the top row shows an affine distorted periodic pattern, its autocorrelation surface, and a set of automatically extracted lattice points. From these, the smallest two translation vectors t_1 and t_2 that generate the pattern lattice are extracted.



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 3: Example of transforming a pattern to square and hexagonal lattices for analysis. See text.

Step 2 involves transforming the lattice to a square grid, aligned with the horizontal and vertical axes. (Figure 3d-f). This is performed by applying an affine transformation to the image and its autocorrelation surface. The transformation used is the unique affine transform leaving the origin (0,0) fixed and taking t_1 to $(L, 0)$ and t_2 to $(0, L)$, where L is the larger of the two generating vectors lengths $||t_1||$ and $||t_2||$. Examining the rotation and reflection symmetries of this new pattern yields the match scores labeled "square" in the table below.

	rot180	rot120	rot90	rot60
square	**0.040**	0.279	0.296	0.269
hexag	**0.040**	**0.038**	0.310	**0.043**
	H refl	V refl	D1 refl	D2 refl
square	0.272	0.275	0.269	0.268
hexag	0.269	0.271	0.271	0.271

These are average squared differences of intensities ranging from 0 to 1, and smaller scores mean that the symmetry is more likely to be present. In this case we find that the pattern only has two-fold rotation symmetry when represented using a square lattice grid. The potential symmetry of this pattern on a square grid is thus *p2*.

Step 3 involves transforming the lattice to a hexagonal grid (Figure 3g-i). This is done by performing the unique affine transformation leaving the origin (0,0) fixed, and mapping t_1 to $(L, 0)$ and t_2 to $(L/2, L*(\sqrt{3}/2))$. The lines labeled "hexag" in the table show rotation and reflection results for the hexagonally transformed pattern. We see that in addition to two-fold symmetry, the pattern now also has 60 and 120 degree rotational symmetry. There are still no reflection symmetries. The potential symmetry of this pattern on a hexagonal grid is classified as *p6*.

Step 4 determines that group *p6* with a hexagonal lattice yields the highest potential symmetry of the pattern, since p2 is a subgroup of p6. Accordingly, from Table 1, we see that this pattern has a two-element symmetry group orbit of ($p2 \rightarrow p6$), which represents all possible symmetry group classifications of this pattern under any affine transformation.

3.1 Shape from Maximal Symmetry

One potential application area of this work is determining local surface orientation from texture under scaled orthographic projection. In particular, we introduce the heuristic that the frontal view of a repeated pattern should be as symmetrical as possible, as determined by the affine orbit of the pattern. In cases where this situation is true, we can then determine the affine transformation that leads to the most symmetrical view of the pattern.

To illustrate, in Figure 4A we have applied an affine warp by hand to a travel photo taken off the web. From this image, the ornamental latticework

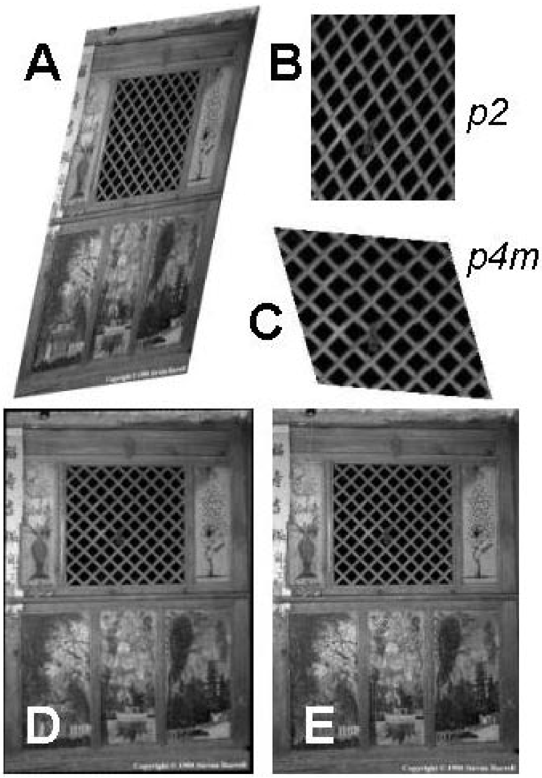


Figure 4: A) Affine distorted image of a doorway. B) Cropped periodic pattern. C) The pattern is unwarped to a square lattice using automatically derived affine transformation F. D) Corresponding unwarped view of the doorway, using transformation F. E) Original image of the doorway, for comparison with D.

was cropped and presented to the wallpaper classification algorithm (4B). The results of potential symmetry classification show that the highest potential symmetry of the pattern occurs with a square lattice (4C). Under these conditions, the pattern has 180 and 90 degree rotational symmetry, and all four reflection symmetries, and the highest potential symmetry of the pattern is the *p4m* wallpaper symmetry group. The orbit of potential symmetries for this pattern under arbitrary affine transformations is ($p2 \rightarrow pmm \rightarrow cmm \rightarrow p4m$). A sample of this orbit, computed using a different pattern, was shown in Figure 1. Finally, since the latticework pattern actually *does* conform to the *p4m* symmetry group in Euclidean space, unwarping the full image by the affine transformation that generated the square lattice results in a frontal view of the original image (4D). The results compare well with the original photo (4E). Determining the highest potential symmetry of this pattern would thus allow us unwarp any unknown affine view of this wall to generate a frontal Euclidean view.

This situation is true for all patterns whose underlying lattice is square or equilateral. After determining the affine transformation that yields a frontal view for such patterns, we can also determine the Euclidean slant and tilt of the plane up to a two-way ambiguity. The slant of a planar surface is the amount of slope (change in depth) of the surface with respect to the viewing direction, and the tilt is the direction in the image of this maximal slope [24]. The important point is that we can compute slant and tilt for a planar surface under scaled orthographic projection by examining the affine transformation that maps the planar wallpaper pattern into a frontal view. Let A be the inverse of this affine transformation, and let the singular value decomposition of $A^T * A$ yield eigenvalues σ_{min} and σ_{max}, with corresponding eigenvectors v_{min} and v_{max}. The slant can then be computed as $\pm\text{acos}(\sigma_{min}/\sigma_{max})$, and tilt as $\text{atan}(v_{min}Y/v_{min}X)$. Note that under affine projection we cannot uniquely determine the sign of the slant.

To test this idea, we artificially generated views of Figure 4 corresponding to different known orientations. For each, we ran the wallpaper extraction algorithm, determined the affine transformation that produces a frontal, square lattice, and from that computed surface slant. The results are listed in the following table:

true slant	10 deg	20 deg	30 deg	40 deg
estimate	10.7	19.7	30.1	39.6
true slant	50 deg	60 deg	70 deg	80 deg
estimate	50.7	45.0	41.2	36.5

We see that slant is well estimated up to 50 degrees. After that point, the warped pattern becomes so distorted that the lattice extraction procedure failed to find the correct lattice. The shape from maximal symmetry approach is novel since, in principle, we could determine the local surface orientation from a single texel, in contrast to approaches that rely on the relative affine transform between neighboring texels [2, 17]. This would allow us to derive structure even when surface curvature is high with respect to the scale of a texel. In practice, however, our current implementation for finding the unit tile (texel) from a wallpaper pattern [15] still requires that we examine multiple neighboring texels.

3.2 Pattern Indexing

Our final example uses an image from the MIT VisTex texture database. Architectural photos are a good source of symmetric patterns. Figure 5A shows two walls of an office building, taken from an unknown viewpoint. We cropped patterns from the left and right side walls by hand (5B). These images represent the same periodic pattern, seen on two surfaces with different orientations. Affine symmetry pattern classification was then performed on both of them. The highest potential symmetry of both patterns is found to be *pmm*, and Figure 5C shows the views after transforming to a square lattice. The similarity of these two views shows the utility of symmetry analysis to normalize patterns for matching and database retrieval. Since the actual pattern on the building surface does not have a square or equilateral lattice, these views are not frontal views, and we cannot compute surface slant. However, the important point of this example is that by normalizing the surface textures to the SAME square lattice, we have rectified them relative to each other and they can thus be properly matched across viewpoints.

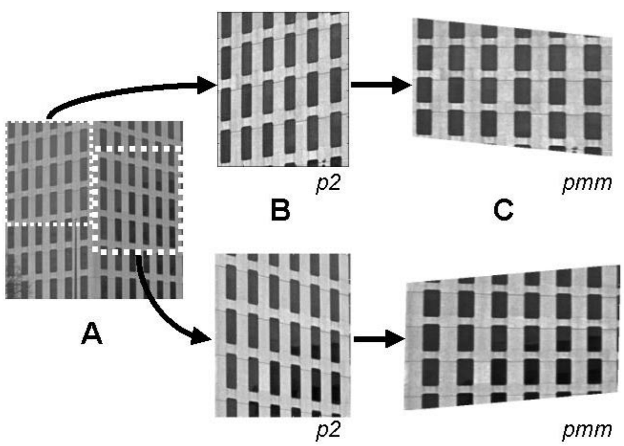


Figure 5: A) Office building photo from the MIT VisTex database. B) Two patterns cropped from A, showing the same pattern on two surfaces of different orientation, C) After normalizing both patterns to have a square lattice, the differences in appearance due to surface orientation are removed.

4 Discussion

In this paper, we have presented a novel idea for regular pattern classification under affine transformation by analyzing skewed symmetry groups. A concise wallpaper symmetry group migration map is presented that separates the relationships of the 17 affinely deformed wallpaper groups into small, distinct orbits. A simple potential symmetry group classification algorithm is shown on both synthetic and real images, with satisfactory results. We have also demonstrated the relevance of this work in regular pattern indexing and textured surface orientation estimation.

Work has been done on finding periodic patterns in real images. In [14] selected image windows are matched with neighboring patches to produce a list of basic elements in the repeated pattern, neighboring patches that match well with these elements, and the affine transformation relating them. More recently, [20] uses image features such as edge, corners or closed contours to find repeating elements. The output is a grouping of the repeating elements (by translations) in the image. These 'repeated pattern' finders are necessary for practical applications, and we could feed the output of these programs to our algorithm for classifying potential symmetry.

One utility of texture analysis is the determination of surface shape. The affine deformation of an observed patch from its most symmetrical potential pattern is a measure of texture deformation that can be used to derive surface orientation [17, 13, 12]. If a periodic pattern has a symmetry group with a square or hexagonal lattice, we can uniquely recover a frontal Euclidean view of the pattern, and the affine deformation that defines the observer's viewing direction with respect to the surface normal. Even in other cases, it may be possible to compute shape from texture using the heuristic that the most symmetrical texture pattern (texel) in an image of a curved surface represents a surface patch perpendicular to the viewing direction. The utility of such apriori knowledge of the orientation of one texel patch is demonstrated by [12].

Our future work will extend the principle of symmetry group classification to patterns viewed in perspective. The first step will be to use cues such as vanishing points to determine the horizon line of the pattern, which can be used to unwarp the image into an affine view of the surface, as shown in [3]. We will also explore applications to curved surfaces where surface geometry will play an important role. Our current emphasis is on the robustness of periodic pattern analysis algorithms to deal with the statistical nature of real images [25].

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