Acceleration Properties of Planar Manipulators from a Dynamic Standpoint

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CMU-RI-TR-88-16

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September 1988

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Abstract

An important issue in designing manipulators for dynamic performance is the determination of the acceleration properties of (some reference-point on) the end-effector of the manipulator. Given the dynamical equations of the planar two degree-of-freedom manipulator and a set of constraints on the actuator torques and on the rates-of-changes of the joint variables, we systematically develop (a) the properties of the linear mapping between the actuator torques and the acceleration of (some reference-point on) the end-effector and (b) the properties of the (non-linear) quadratic mapping between the rates-of-changes of the joint variables and the acceleration of the end-effector. We then show how these mappings can be combined to obtain useful acceleration sets - for example the acceleration set corresponding to any point in the workspace of the manipulator - as well as the properties of these sets.

1 Introduction

An important issue in designing manipulators for dynamic performance is the determination of the acceleration properties of (some reference-point on) the end-effector of the manipulator. Given the dynamical equations of the planar two degree-of-freedom manipulator and a set of constraints on the actuator torques and on the rates-of-changes of the joint variables, we systematically develop (a) the properties of the linear mapping between the actuator torques and the acceleration of (some reference-point on) the end-effector and (b) the properties of the (non-linear) quadratic mapping between the rates-of-changes of the joint variables and the acceleration of the end-effector. We then show how these mappings can be combined to obtain useful acceleration sets - for example the acceleration set corresponding to any point in the workspace of the manipulator - as well as the properties of these sets.

It is useful to briefly mention the work done by others on related problems. Khatib [1, 2] sets up an optimization problem to improve the acceleration of the end-effector; however the non-linear terms in the dynamical equations are accounted for in somewhat ad-hoc fashion (by taking certain "high" and "low" values of these terms). Graettinger and Krogh [3] use semi-infinite programming to obtain the "acceleration-radius" (or "isotropic acceleration") of a manipulator. In contrast to our approach, both the above-mentioned approaches do not yield the acceleration sets of the manipulators (defined in section 2) or the properties of these sets or show how the properties of the sets are related to the geometric and dynamical parameters of the manipulator.

The report is organized as follows: The problem and the variables of interest (in the problem) are defined in section 2. In the next section, we describe certain decompositions of the manipulator Jacobians which are useful in deriving certain properties of the linear and (non-linear) quadratic mappings. The properties of the linear mapping and the quadratic mapping, respectively, are derived in section 4 and 5. In section 6, we show how these maps can be combined to obtained the acceleration set corresponding to any point in the state space; we then determine the properties of this set. Similarly, section 7 is devoted to determining the acceleration set (and its properties) corresponding to any configuration (or "position") in the workspace of the manipulator. In the final section, using the example of the planar manipulator in our laboratory, we address the computation of the acceleration sets and their properties.

2 Definition of the Problem

In this section, using a simple two degree-of-freedom manipulator, we define the problem to understand the dynamic performance of manipulators.

First, we define manipulator variables in subsection 2.1. In the following subsection, we express the manipulator acceleration in terms of the variables. In subsection 2.3, we define the problem to characterize the manipulator performance with end-effector acceleration.

2.1 Definition of the manipulator variables

Consider the serial two degree-of-freedom manipulator with two revolute joints shown in Figure 1. The manipulator is assumed to be rigid with negligible joint friction.

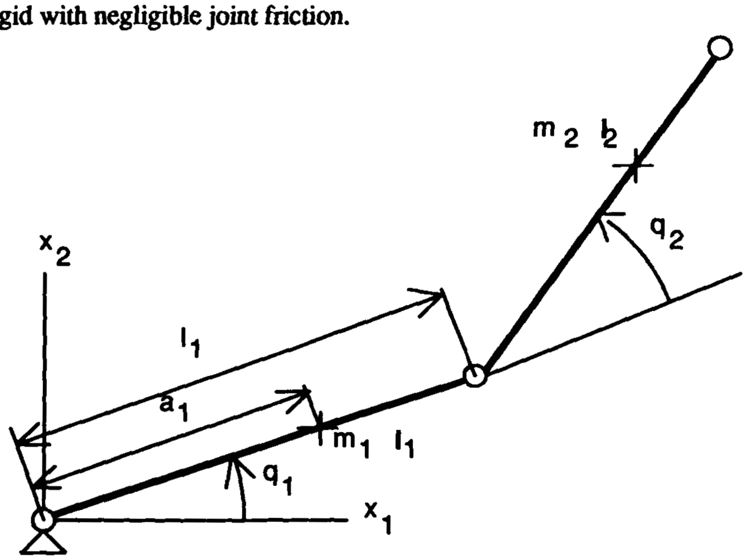

Figure 1: A two degree-of-freedom manipulator

Let l_1 denote the length of link 1, a_1 the distance from joint axis 1 to the center of mass of link 1, m_1 the mass of link 1 and I_1 the principal moment of inertia of link 1 with respect to its center of mass about an axis perpendicular to the plane of the motion. Similarly, let l_2, a_2, m_2 and I_2 denote the corresponding properties of link 2 (See Figure 1). We call these quantities design variables.

τ_1 and τ_2 denote the joint torques, respectively, at joints 1 and 2 and

$$\tau = [\tau_1 \ \ \tau_2]^T \tag{1}$$

denotes the vector of joint torque vectors. Let

$$|\tau_i| \le \tau_{io}, \ i=1,2 \tag{2}$$

denote the constraints on the actuator torques at joints 1 and 2. We define

$$T = \{\tau \mid |\tau_i| \le \tau_{io}, \ i=1,2\} \tag{3}$$

to be the set of allowable torques.

Let (x_1,x_2) denote the coordinates, in a reference frame fixed to the base, of a reference point P on link 2 (See

Figure 1) and define

$$\mathbf{x} = [x_1 \;\; x_2]^T \tag{4}$$

to be the vector of task coordinates in task space. Let q_1 and q_2 denote the generalized coordinates of the manipulator (See Figure 1), q_1 being the joint variable at joint 1 and q_2 the joint variable at joint 2. Define

$$\mathbf{q} = [q_1 \;\; q_2]^T \tag{5}$$

to be the vector of joint variables in joint space. If

$$q_{iL} \leq q_i \leq q_{iU}, \;\; i=1,2 \tag{6}$$

denotes the constraint on joint variable i, then we can define the workspace W of a manipulator as

$$W = \{\mathbf{q} \mid q_{iL} \leq q_i \leq q_{iU}, \;\; i=1,2\} \tag{7}$$

Let $\dot{q}_1$ and $\dot{q}_2$ denote the joint velocities (the rates of change of the joint variables) q_1 and q_2, respectively. Define

$$\dot{\mathbf{q}} = [\dot{q}_1 \;\; \dot{q}_2]^T \tag{8}$$

to be the vector of joint velocities. If

$$|\dot{q}_i| \leq \dot{q}_{io}, \;\; i=1,2 \tag{9}$$

denotes the constraints on the rates of changes of the joint variables, then we can define

$$F = \{\dot{\mathbf{q}} \mid |\dot{q}_i| \leq \dot{q}_{io}, \;\; i=1,2\} \tag{10}$$

to be the set of all possible joint velocity vectors.

Define the state vector

$$\mathbf{u} := (\mathbf{q} \;\; \dot{\mathbf{q}}) = [q_1 \;\; q_2 \;\; \dot{q}_1 \;\; \dot{q}_2]^T \tag{11}$$

to represent the dynamic state of a manipulator in the state-space.

The acceleration space or acceleration plane, A, of the manipulator is the set of all possible accelerations, $\ddot{\mathbf{x}} = [\, \ddot{x}_1 \;\; \ddot{x}_2 \,]^T$, where $\ddot{x}_1$ and $\ddot{x}_2$ are real numbers.

More formally we can define

$$A = \{\ddot{\mathbf{x}} \mid \ddot{\mathbf{x}} \in R^2\} \tag{12}$$

where R^2 is the real Euclidean plane.

2.2 Manipulator acceleration

In this subsection, we derive an expression for the acceleration [4],

$$\ddot{\mathbf{x}} = [\ddot{x}_1 \;\; \ddot{x}_2]^T, \tag{13}$$

of the reference-point P on the end-effector, since this quantity plays an important role in our analysis.

The relationship between the velocity, $\dot{\mathbf{x}}$, of point P, and the "joint velocity" vector $\dot{\mathbf{q}}$ is well known:

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}} \tag{14}$$

where **J** is a (2×2) matrix called the manipulator Jacobian. The detailed expression of Jacobian matrix is shown in Appendix.

The dynamic behavior of the two degree-of-freedom manipulator is described by the following equations:

$$d_{11}(q_2)\ddot{q}_1 + d_{12}(q_2)\ddot{q}_2 - v(q_2)(\dot{q}_1^2+2\dot{q}_1\dot{q}_2) + p_1(\mathbf{q}) = \tau_1 \tag{15}$$

$$d_{21}(q_2)\ddot{q}_1 + d_{22}\ddot{q}_2 - v(q_2)(\dot{q}_1^2) + p_2(\mathbf{q}) = \tau_2 \tag{16}$$

where the coefficients $d_{ij}(q_2)$, (i,j=1,2), $v(q_2)$, $p_i(\mathbf{q})$, (i = 1,2) are given in Appendix; note that coefficients d_{ij} (i,j =1,2), $v(q_2)$ are functions of only joint variable q_2.

Defining

$$\mathbf{D}(q_2) := \begin{bmatrix} d_{11}(q_2) & d_{12}(q_2) \\ d_{21}(q_2) & d_{22} \end{bmatrix}, \tag{17}$$

$$\mathbf{V}(q_2) := \begin{bmatrix} 0 & -v(q_2) \\ v(q_2) & 0 \end{bmatrix}, \tag{18}$$

$$\mathbf{p}(\mathbf{q}) := [p_1(\mathbf{q}) \;\; p_2(\mathbf{q})]^T \tag{19}$$

$$\ddot{\mathbf{q}} := [\ddot{q}_1 \;\; \ddot{q}_2]^T \tag{20}$$

and

$$\dot{\mathbf{q}}^2 := [\dot{q}_1^2 \;\; (\dot{q}_1+\dot{q}_2)^2-\dot{q}_1^2]^T \tag{21}$$

the dynamic equations (15) and (16) become,

$$\mathbf{D}(q_2)\ddot{\mathbf{q}} + \mathbf{V}(q_2)\{\dot{\mathbf{q}}\}^2 + \mathbf{p}(\mathbf{q}) = \tau \tag{22}$$

The matrix $\mathbf{D}(q_2)$ is the mass matrix of the manipulator and the vector $\mathbf{p}(\mathbf{q})$ denotes the gravitational terms influencing the dynamic behavior.

A crucial step in the acceleration analysis is the definition of the skew-symmetric matrix $\mathbf{V}(q_2)$ and the vector $\{\dot{\mathbf{q}}\}^2$, which allows all the non-linear terms (often called Coriolis and centrifugal) to be written as the product of $\mathbf{V}(q_2)$ and $\{\dot{\mathbf{q}}\}^2$. The notation $\{\;\}^2$ is used to draw attention to the fact that the elements of the vector $\{\dot{\mathbf{q}}\}^2$ are quadratic in the rates-of- changes $\dot{q}_1$ and $\dot{q}_2$, respectively, of the joint variables q_1 and q_2. Note that $\{\dot{\mathbf{q}}\}^2$ is not equal to $\dot{\mathbf{q}}^2=\dot{q}_1^2+\dot{q}_2^2$.

To obtain the expression for the acceleration $\ddot{\mathbf{x}}$ of the point P, we differentiate (14) to obtain

$$\ddot{\mathbf{x}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} \tag{23}$$

In Appendix, we show that the second term in (23), $\dot{\mathbf{J}}\,\dot{\mathbf{q}}$, can be written in the form

$$\dot{\mathbf{J}}\dot{\mathbf{q}} = -\mathbf{E}(\mathbf{q})\{\dot{\mathbf{q}}\}^2 \tag{24}$$

Combining (23) and (24) we obtain

$$\ddot{\mathbf{x}} = \mathbf{J}\ddot{\mathbf{q}} - \mathbf{E}(\mathbf{q})\{\dot{\mathbf{q}}\}^2. \tag{25}$$

Defining the quantities,

$$\mathbf{A}(\mathbf{q}) = \mathbf{J}(\mathbf{q})\mathbf{D}^{-1}(q_2), \tag{26}$$

$$\mathbf{B(q)} = -\mathbf{A(q)V(q)} - \mathbf{E(q)}, \tag{27}$$

$$\mathbf{c(q)} = -\mathbf{A(q)p(q)}, \tag{28}$$

it is easy to verify that the expression for the acceleration $\ddot{\mathbf{x}}$ of the point P, obtained by combining equation (22) with equations (25) through (28), is given by

$$\ddot{\mathbf{x}} = \mathbf{A(q)}\tau + \mathbf{B}\{\dot{\mathbf{q}}\}^2 + \mathbf{c(q)}. \tag{29}$$

Note that $\mathbf{A(q)}$, $\mathbf{B(q)}$ and $\mathbf{c(q)}$ are position dependent, the expressions for the coefficients of which are given in Appendix.

If the manipulator operates in a (horizontal) plane perpendicular to gravity, then $\mathbf{c(q)} = \mathbf{0}$ and (29) becomes

$$\ddot{\mathbf{x}} = \mathbf{A(q)}\tau + \mathbf{B}\{\dot{\mathbf{q}}\}^2. \tag{30}$$

In this paper, we will study manipulators, moving in horizontal planes, whose acceleration properties are described by equation (30).

Defining

$$\alpha_\tau := [\alpha_{1\tau} \ \ \alpha_{2\tau}]^T := \mathbf{A(q)}\tau \tag{31}$$

,and

$$\alpha_{\dot{q}} := [\alpha_{1\dot{q}} \ \ \alpha_{2\dot{q}}]^T := \mathbf{B(q)}\{\dot{\mathbf{q}}\}^2 \tag{32}$$

equation (30) can be written as

$$\ddot{\mathbf{x}} = \alpha_\tau + \alpha_{\dot{q}} \tag{33}$$

It is convenient to think of α_τ as the contribution of the torques to the acceleration of the reference point P and $\alpha_{\dot{q}}$ as the contribution of the joint-rates to the acceleration of P, the sum of these two quantities giving us the acceleration of P as expressed by equation (33).

Equation (31) can be viewed as a linear, position-dependent, mapping between the torque vector τ and its contribution α_τ to the acceleration of P. Similarly equation (32) can be viewed as a quadratic, position-dependent, mapping between the joint rate vector $\dot{\mathbf{q}}$ and its contribution $\alpha_{\dot{q}}$ to the acceleration of P.

2.3 Definition of the problem

The acceleration $\ddot{\mathbf{x}}$ of the reference point P of a manipulator, specified by its design variables, constraints on the torques as given by (2) or (3), constraints on the joint variables as given by (6) or (7) and constraints on the joint velocities as given by (9) or (10), will be a subset of the acceleration plane A of equation (12). In other words, the acceleration set for a combination of the above constraints can represent the dynamic performance of manipulators. To characterize the manipulator dynamic performance, we generate four acceleration sets as follows:

First, we consider the manipulator acceleration set when the joint velocity is zero. Physically, the set represents the manipulator dynamics when a manipulator starts to move. For the given set T of allowable actuator torques described by (3), we define the set of all allowable α_τ as

$$\mathrm{S}_\tau = \{\alpha_\tau \mid (\exists\tau \in \mathrm{T})(\alpha_\tau = \mathrm{A}\tau)\} \tag{34}$$

Next, when the actuator torque vanishes during the operation of a manipulator, the subsequent motion of a manipulator is also critical in manipulator dynamics. For the given (constraint) set F of allowable rates-of-change, described by (10), we define the set $S_{\dot{\mathbf{q}}}$ of all allowable $\alpha_{\dot{q}}$ as

$$S_{\dot{q}} = \{\alpha_{\dot{q}} \mid (\exists \dot{\mathbf{q}} \in F)(\alpha_{\dot{q}} = \mathbf{B}\{\dot{\mathbf{q}}\}^2)\} \tag{35}$$

Finally, when a manipulator is in motion, we consider two acceleration sets in the following. The acceleration of the reference point P corresponding to a specified state-vector $\mathbf{u} = [\, q_1, q_2, \dot{q}_1, \dot{q}_2 \,]^T$ in the state-space will be denoted by $\alpha_{\mathbf{u}}$. From equation (30), we write

$$\alpha_{\mathbf{u}} = \mathbf{A}(\mathbf{q})\tau + \mathbf{B}(\mathbf{q})\{\dot{\mathbf{q}}\}^2 \tag{36}$$

If we define a constant vector **k**,

$$\mathbf{k}(\mathbf{u}) = \mathbf{k}(\mathbf{q},\dot{\mathbf{q}}) = [k_1,\ k_2]^T = \alpha_{\dot{\mathbf{q}}}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{B}(\mathbf{q})\{\dot{\mathbf{q}}\}^2, \tag{37}$$

then (36) can be written as

$$\alpha_{\mathbf{u}} = \mathbf{A}(\mathbf{q})\tau + \mathbf{k}(\mathbf{u}) \tag{38}$$

We now define the acceleration set, $S_{\mathbf{u}}$, at a specified point **u** in the state space as follows: For a given set T of allowable actuator torques described by (3), the acceleration set $S_{\mathbf{u}}$ at a point $\mathbf{u} = [\, \mathbf{q}, \dot{\mathbf{q}} \,]^T$ in the state - space is given by

$$S_{\mathbf{u}}(\mathbf{q},\dot{\mathbf{q}}) = \{\alpha_{\mathbf{u}} \mid (\mathbf{u}=(\mathbf{q},\dot{\mathbf{q}})),\ (\exists\tau \in T)(\tau_{\mathbf{u}}=\mathbf{A}(\mathbf{q})\tau+\mathbf{k}(\mathbf{u}))\} \tag{39}$$

Thus $S_{\mathbf{u}}$ is the image of the set T under the mapping (38).

Finally, at a given position $\mathbf{q} = [\, q_1, q_2 \,]^T$ in the workspace of the manipulator, we can define two sets

$$(S_L)_1 = \cup_{\dot{\mathbf{q}} \in F} S_{\mathbf{u}}(\mathbf{q},\dot{\mathbf{q}}) \tag{40}$$

$$(S_L)_2 = \cap_{\dot{\mathbf{q}} \in F} S_{\mathbf{u}}(\mathbf{q},\dot{\mathbf{q}}) \tag{41}$$

The supremum of $(S_L)_1$ will be give us the magnitude of the maximum acceleration (in some direction) of the reference point P at a given position (q_1, q_2) of the manipulator.

The infimum of $(S_L)_2$ will give us the magnitude of the maximum acceleration of the reference point P available in all direction at a given position of the manipulator. The infimum of $(S_L)_2$ is called the isotropic acceleration in Khatib [1] and the local acceleration radius in Kim [5].

Based on the above definitions, the manipulator problem can be written as follows;

1. To characterize four acceleration sets, S_τ, $S_{\dot{q}}$, $S_{\mathbf{u}}$, S_L, with their shape and supremum and infimum.
2. To examine the behavior of four acceleration sets S_τ, $S_{\dot{q}}$, $S_{\mathbf{u}}$, S_L.

In Section 4 and 5, respectively, we study the properties of the linear mapping described by equation (31) and the properties of the non-linear mapping described by equation (32). The acceleration properties of P, obtained by combining these two maps, is discussed in Section 6. In the next section we present a decomposition of the Jacobian which is helpful in the study of the aforementioned linear and non-linear maps.

3 Decomposition of the manipulator jacobian

In this section, we derive some little known decompositions of the manipulator Jacobians. These decompositions facilitate the derivation of the properties of the acceleration properties in Sections 4 and 5. Let v denote the velocity of point P in a reference frame fixed to the base N, ($\mathbf{n}_1$ $\mathbf{n}_2$) be a set of dextral orthogonal unit vectors fixed in N, ($\mathbf{a}_1$ $\mathbf{a}_2$) a set of dextral orthogonal unit vectors fixed in A, and ($\mathbf{b}_1$ $\mathbf{b}_2$) a set of dextral orthogonal vectors fixed in B (see Figure 2).

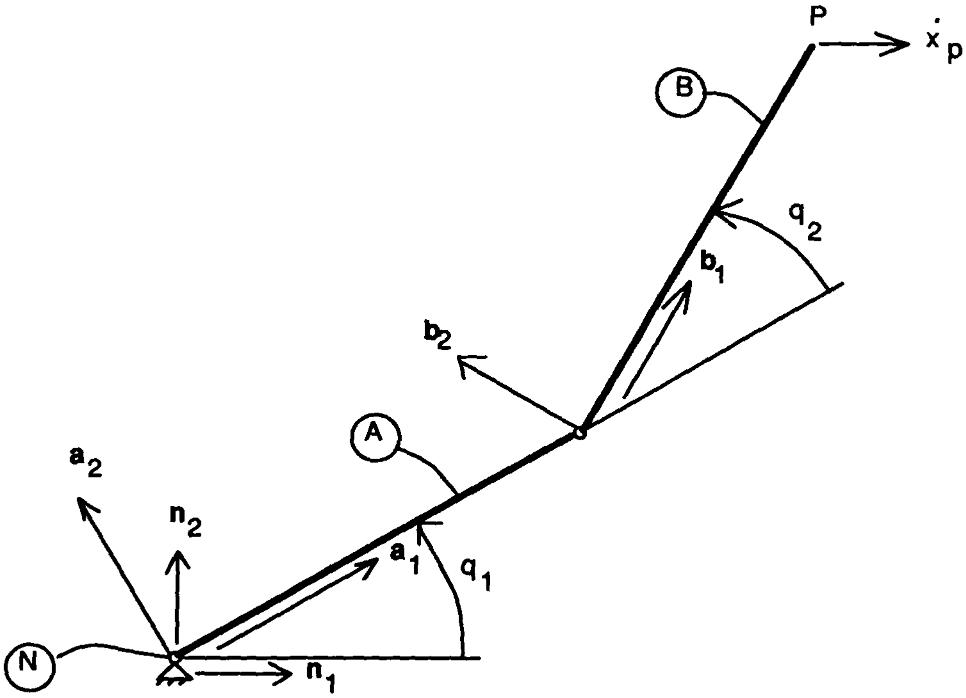

Figure 2: Description of the velocity vector.

We pick $\mathbf{n}_1$ so that it points in the direction of the positive x_1 axis and pick $\mathbf{n}_2$ so that it points in the direction of the positive x_2 axis (Figure 2).

Defining,

$$\mathbf{R}(q_i) := \begin{bmatrix} \cos q_i & -\sin q_i \\ \sin q_i & \cos q_i \end{bmatrix}, \quad (i = 1,2) \tag{42}$$

and referring to Figure 2, we obtain,

$$[\mathbf{a}_1 \ \ \mathbf{a}_2]^T = \mathbf{R}(q_2) \ [\mathbf{b}_1 \ \ \mathbf{b}_2]^T \tag{43}$$

$$[\mathbf{n}_1 \ \ \mathbf{n}_2]^T = \mathbf{R}(q_1) \ [\mathbf{a}_1 \ \ \mathbf{a}_2]^T. \tag{44}$$

The velocity of the reference point,v, in the reference frame N is given by

$$\mathbf{v} = l_1\dot{q}_1\mathbf{a}_2 + l_2(\dot{q}_1+\dot{q}_2)\mathbf{b}_2 \tag{45}$$

From equation (45), we obtain

$$\mathbf{v}\bullet\mathbf{b}_1 = l_1\dot{q}_1\mathbf{a}_2\bullet\mathbf{b}_1 \tag{46}$$

$$\mathbf{v}\bullet\mathbf{b}_2 = l_1\dot{q}_1\mathbf{a}_2\bullet\mathbf{b}_2 + l_2(\dot{q}_1+\dot{q}_2). \tag{47}$$

Note that $\mathbf{v} \bullet \mathbf{b}_1$ and $\mathbf{v} \bullet \mathbf{b}_2$ are simply the $\mathbf{b}_1$ and $\mathbf{b}_2$ measure numbers of the velocity of P in N. Using equations (42), equations (46) and (47) can be rewritten in the matrix form,

$$\begin{bmatrix} \mathbf{v}\bullet\mathbf{b}_1 \\ \mathbf{v}\bullet\mathbf{b}_2 \end{bmatrix} \quad \begin{bmatrix} l_1\sin q_2 & 0 \\ l_1\cos q_2+l_2 & l_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}. \tag{48}$$

Defining,

$$\mathbf{M}(q_2) := \begin{bmatrix} l_1\sin q_2 & 0 \\ & \\ l_1\cos q_2+l_2 & l_2 \end{bmatrix}, \tag{49}$$

and using (5), equation (48) becomes

$$[\mathbf{v}\bullet\mathbf{b}_1 \;\; \mathbf{v}\bullet\mathbf{b}_2]^T = \mathbf{M}(q_2)\ \dot{\mathbf{q}}. \tag{50}$$

Referring again to the geometry of Figure 2, we can write,

$$[\mathbf{v}\bullet\mathbf{a}_1 \;\; \mathbf{v}\bullet\mathbf{a}_2]^T = \mathbf{R}(q_2)\ [\mathbf{v}\bullet\mathbf{b}_1 \;\; \mathbf{v}\bullet\mathbf{b}_2]^T \tag{51}$$

$$[\mathbf{v}\bullet\mathbf{n}_1 \;\; \mathbf{v}\bullet\mathbf{n}_2]^T = \mathbf{R}(q_1)\ [\mathbf{v}\bullet\mathbf{a}_1 \;\; \mathbf{v}\bullet\mathbf{a}_2]^T \tag{52}$$

Equation (51) simply relates the $\mathbf{b}_1$ and $\mathbf{b}_2$ measure numbers of $\mathbf{v}$ to the $\mathbf{a}_1$ and $\mathbf{a}_2$ measure numbers of $\mathbf{v}$; similarly (52) relates the $\mathbf{a}_1$ and $\mathbf{a}_2$ measure numbers of $\mathbf{v}$ to the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of $\mathbf{v}$.

Combining equations (50) through (52) we obtain

$$[\mathbf{v}\bullet\mathbf{n}_1 \;\; \mathbf{v}\bullet\mathbf{n}_2] = \mathbf{R}(q_1)\ \mathbf{R}(q_2)\ \mathbf{M}(q_2)\ \dot{\mathbf{q}} \tag{53}$$

If $\mathbf{x}$ is the position vector of P with respect to the fixed point pivot O, then from equation (4) and the choice of the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$ (see Figure 2), we can write

$$\mathbf{x} = x_1\mathbf{n}_1 + x_2\mathbf{n}_2 \tag{54}$$

and,

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{x}_1\mathbf{n}_1 + \dot{x}_2\mathbf{n}_2 \tag{55}$$

From (55), the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of $\mathbf{v}$ are given by

$$[\mathbf{v}\bullet\mathbf{n}_1 \;\; \mathbf{v}\bullet\mathbf{n}_2]^T = [\dot{x}_1 \;\; \dot{x}_2]^T \tag{56}$$

Combining (53) and (56) we obtain,

$$\dot{\mathbf{x}} = [\dot{x}_1 \;\; \dot{x}_2]^T = \mathbf{R}(q_1)\ \mathbf{R}(q_2)\ \mathbf{M}(q_2)\ \dot{\mathbf{q}} \tag{57}$$

Comparing equations (57) and (14), we obtain

$$\mathbf{J}(\mathbf{q}) = \mathbf{R}(q_1)\ \mathbf{R}(q_2)\ \mathbf{M}(q_2) \tag{58}$$

Note that we have decomposed the Jacobian matrix, $\mathbf{J}(\mathbf{q})$, into the product of three matrices which depend either on q_1 or q_2 but not both q_1 and q_2. $\mathbf{R}(q_1)$ and $\mathbf{R}(q_2)$ are simple orthogonal matrices and $\mathbf{N}(q_2)$ represents the kinematic coupling between the two links. In a similar fashion, we can show that the matrix $\mathbf{E}(\mathbf{q})$ in equation (24) can be written as

$$\mathbf{E}(\mathbf{q}) = \mathbf{R}(q_1)\ \mathbf{R}(q_2)\ \mathbf{N}(q_2) \tag{59}$$

where

$$\mathbf{N}(q_2) := \begin{bmatrix} l_1 \cos q_2 + l_2 & l_2 \\ -l_1 \sin q_2 & 0 \end{bmatrix}. \tag{60}$$

Equations (58) and (59) are very useful in deriving the properties of the acceleration maps in section 4 and 5.

4 Linear mapping

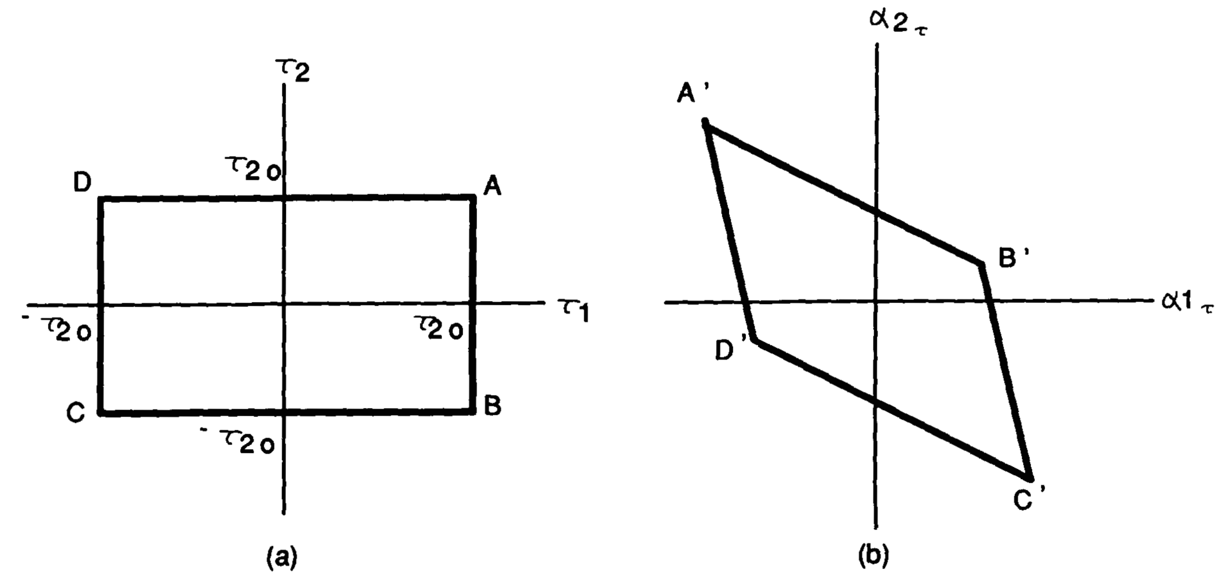

Figure 3: Linear Mapping

For the given set T of allowable actuator torques described by (3) and represented by (the interior and the boundary of) the rectangle ABCD shown in Figure 3-a, we defined the set of all allowable α_τ in section 2 as

$$S_\tau = \{\alpha_\tau \mid (\exists\tau \in T)(\alpha_\tau = A\tau)\}.$$

In this section, we simply state that S_τ is the parallelogram $A'B'C'D'$ shown in Figure 3-b and we will derive three properties of the set S_τ. The first property determines the "size" of S_τ, as characterized by its infimum, supremum, and its area. The second property expresses the q_1 invariance of S_τ. The third property expresses the q_2 dependency of S_τ.

The decomposition, expressed by equation (58), of the manipulator Jacobian is extremely useful in the derivation of the properties of the set S_τ. Combining equation (58) and (26) we obtain

$$\mathbf{A(q)} = \mathbf{R}(q_1)\ \mathbf{R}(q_2)\ \mathbf{M}(q_2)\ \mathbf{D}^{-1}(q_2). \tag{61}$$

Defining a matrix $\mathbf{P(q_2)}$ as

$$\mathbf{P}(q_2) = \mathbf{R}(q_2)\ \mathbf{M}(q_2)\ \mathbf{D}^{-1}(q_2), \tag{62}$$

we can rewrite (61) as

$$\mathbf{A(q)} = \mathbf{R}(q_1)\ \mathbf{P}(q_2). \tag{63}$$

Note that we have decomposed $\mathbf{A(q)}$ into an orthogonal matrix $\mathbf{R}(q_1)$ depending only on q_1 and a matrix $\mathbf{P}(q_2)$ depending only on q_2. This decomposition facilitates the derivation of the properties below. The (i,j) element of **A** and **P** will be denoted, respectively, by a_{ij}, and p_{ij}.

Let A', B', C', D' denote, respectively, the points in the α_τ - plane into which the points $A(\tau_{1o}, \tau_{2o})$, $B(\tau_{1o}, -\tau_{2o})$, $C(-\tau_{1o}, -\tau_{2o})$, and $D(-\tau_{1o}, \tau_{2o})$ map. Then it is easy to verify that the coordinates of A', B', C', D' in the α_τ -

plane are given by

$$A' \ (a_{11}\tau_{1o}+a_{12}\tau_{2o} \ \ a_{21}\tau_{1o}+a_{22}\tau_{2o}) \tag{64}$$

$$B' \ (a_{11}\tau_{1o}-a_{12}\tau_{2o} \ \ a_{21}\tau_{1o}-a_{22}\tau_{2o}) \tag{65}$$

$$C' \ (-a_{11}\tau_{1o}-a_{12}\tau_{2o} \ \ -a_{21}\tau_{1o}-a_{22}\tau_{2o}) \tag{66}$$

$$D' \ (-a_{11}\tau_{1o}+a_{12}\tau_{2o} \ \ -a_{21}\tau_{1o}+a_{22}\tau_{2o}). \tag{67}$$

Since (31) is a linear mapping, A′B′C′D′ is a parallelogram; the equations of the sides A′B′, B′C′, C′D′ and D′A′ are readily obtained as

$$A'B' \quad \frac{1}{a_{12}}a_{1\tau} - \frac{1}{a_{22}}a_{2\tau} + \left(-\frac{a_{11}}{a_{12}}+\frac{a_{21}}{a_{22}}\right)\tau_{1o} = 0, \ (a_{11}\tau_{1o}-a_{12}\tau_{2o} \le a_{1\tau} \le a_{11}\tau_{1o}+a_{12}\tau_{2o}) \tag{68}$$

$$B'C' \quad \frac{1}{a_{11}}a_{1\tau} - \frac{1}{a_{21}}a_{2\tau} + \left(-\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)\tau_{2o} = 0, \ (-a_{11}\tau_{1o}+a_{12}\tau_{2o} \le a_{1\tau} \le a_{11}\tau_{1o}+a_{12}\tau_{2o}) \tag{69}$$

$$C'D' \quad \frac{1}{a_{12}}a_{1\tau} - \frac{1}{a_{22}}a_{2\tau} - \left(-\frac{a_{11}}{a_{12}}+\frac{a_{21}}{a_{22}}\right)\tau_{1o} = 0, \ (-a_{11}\tau_{1o}-a_{12}\tau_{2o} \le a_{1\tau} \le -a_{11}\tau_{1o}+a_{12}\tau_{2o}) \tag{70}$$

$$D'A' \quad \frac{1}{a_{11}}a_{1\tau} - \frac{1}{a_{21}}a_{2\tau} - \left(-\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)\tau_{2o} = 0, \ (-a_{11}\tau_{1o}-a_{12}\tau_{2o} \le a_{1\tau} \le a_{11}\tau_{1o}-a_{12}\tau_{2o}). \tag{71}$$

•**Property 1**: The set S_τ is a parallelogram whose supremum, infimum and area are given by

$$\sup(S_\tau) = \max\left[\sqrt{(p_{11}^2+p_{21}^2)\tau_{1o}^2+(p_{12}^2+p_{22}^2)\tau_{2o}^2 \pm 2(p_{11}p_{12}+p_{21}p_{22})\tau_{1o}\tau_{2o}} \ \right]. \tag{72}$$

$$\inf(S_\tau) = \min\left[\frac{|(p_{11}p_{22}-p_{12}p_{21}) \ \tau_{1o}|}{\sqrt{p_{12}^2+p_{22}^2}} \quad \frac{|(p_{11}p_{22}-p_{12}p_{21}) \ \tau_{2o}|}{\sqrt{p_{11}^2+p_{21}^2}} \right] \tag{73}$$

$$\textbf{area}(S_\tau) = \left| \frac{l_1 l_2 \sin q_2}{(I_1+m_1a_1^2)(I_2+m_2a_2^2)+(I_2+m_2a_2^2\sin^2 q_2)m_2 l_1^2} \right| 4\tau_{1o}\tau_{2o} \tag{74}$$

•**Proof**:

If σ(O′A′), σ(O′B′), σ(O′C′), σ(O′D′) denote, respectively, the distances of the vertices A′, B′, C′, and D′ from the origin O′, then the supremum of S_τ is given by

$$\sup(S_\tau) = \max[\sigma(O'A'), \ \sigma(O'B'), \ \sigma(O'C'), \ \sigma(O'D')] \tag{75}$$

Since A′ and C′ are equidistant from the origin O′ and since B′ and D′ are also equidistant from the origin, (observation (1) above),

$$\sup(S_\tau) = \max[\sigma(O'A'), \ \sigma(O'B')] = \frac{1}{2}\max[\sigma(A'C'), \ \sigma(B'D')] \tag{76}$$

where σ(A′C′) and σ(B′D′) denote the lengths of the diagonals A′C′ and B′D′. Combining equations (64) - (67), (63) and (42), σ(A′C′) and σ(B′D′) are given by

$$\sigma(A'C') = \sqrt{(p_{11}^2+p_{21}^2)\tau_{1o}^2+(p_{12}^2+p_{22}^2)\tau_{2o}^2+2(p_{11}p_{12}+p_{21}p_{22})\tau_{1o}\tau_{2o}} \tag{77}$$

$$\sigma(B'D') = \sqrt{(p_{11}^2+p_{21}^2)\tau_{1o}^2+(p_{12}^2+p_{22}^2)\tau_{2o}^2-2(p_{11}p_{12}+p_{21}p_{22})\tau_{1o}\tau_{2o}} \tag{78}$$

Therefore,

$$\sup(S_\tau) = \max\left[\sqrt{(p_{11}^2+p_{21}^2)\tau_{1o}^2+(p_{12}^2+p_{22}^2)\tau_{2o}^2 \pm 2(p_{11}p_{12}+p_{21}p_{22})\tau_{1o}\tau_{2o}} \ \right]. \tag{79}$$

If ρ(O′A′), ρ(O′B′), ρ(O′C′), ρ(O′D′) denote, respectively, the distances from O′ to A′B′, B′C′, C′D′, D′A′, then the infimum of S_τ is given by

$$\inf(S_\tau) = \min[\rho(O'A'),\ \rho(O'B'),\ \rho(O'C'),\ \rho(O'D')] \tag{80}$$

Since A'B' and C'D' are equidistant from O' and since B'C' and D'A' are also equidistant from O', (observation (2) above)

$$\inf(S_\tau) = \min[\rho(A'B'),\ \rho(B'C')] \tag{81}$$

The distance ρ from the origin to a general line $\alpha x + \beta y + \chi = 0$ in the xy - plane is given by

$$\rho = \frac{|\chi|}{\sqrt{\alpha^2 + \beta^2}}. \tag{82}$$

Substituting appropriate values of α, β, and χ from equations (68) and (69) into equation (82) and using equations (63) and (42), we obtain

$$\rho(A'B') = \frac{|(p_{11}p_{22}-p_{12}p_{21})\ \tau_{1o}|}{\sqrt{p_{12}^2+p_{22}^2}} = \frac{|\det\mathbf{P}(q_2)|\tau_{1o}}{\sqrt{p_{12}^2+p_{22}^2}} \tag{83}$$

$$\rho(B'C') = \frac{|(p_{11}p_{22}-p_{12}p_{21})\ \tau_{2o}|}{\sqrt{p_{11}^2+p_{21}^2}} = \frac{|\det\mathbf{P}(q_2)|\tau_{2o}}{\sqrt{p_{11}^2+p_{21}^2}} \tag{84}$$

Substituting (83) and (84) into (81), we obtain

$$\inf(S_\tau) = \min\left(\frac{|\det\mathbf{P}(q_2)|\tau_{1o}}{\sqrt{p_{12}^2+p_{22}^2}}\quad \frac{|\det\mathbf{P}(q_2)|\tau_{2o}}{\sqrt{p_{11}^2+p_{21}^2}}\right) \tag{85}$$

The $\det\mathbf{P}(q_2)$ vanishes at $q_2 = 0, \pi$. Therefore $\inf(S_\tau) = 0$ at $q_2=0, \pi$.

Since the area transformed by the linear mapping **A** is

$$\int_{-\tau_{1o}}^{\tau_{1o}}\int_{-\tau_{2o}}^{\tau_{2o}} \det(\mathbf{A})d\tau_1 d\tau_2, \tag{86}$$

the area of the parallelogram is obtained as

$$\det(\mathbf{A})\ [\tau_{1o}-(-\tau_{1o})]\ [\tau_{2o}-(-\tau_{2o})] \tag{87}$$

$$= \det(\mathbf{R}(q_1))\ \det(\mathbf{R}(q_2))\ \det(\mathbf{M}(q_2))\ \det(\mathbf{D}^{-1}(q_1))\ 4\tau_{1o}\tau_{2o} \tag{88}$$

$$= \det(\mathbf{M}(q_2))\ \det(\mathbf{D}^{-1}(q_1))\ 4\tau_{1o}\tau_{2o} \tag{89}$$

$$= \left| \frac{l_1 l_2 \sin q_2}{(I_1+m_1a_1^2)(I_2+m_2a_2^2)+(I_2+m_2a_2^2\sin^2 q_2)m_2 l_1^2} \right| 4\tau_{1o}\tau_{2o} \tag{90}$$

•**Property 2:** q_1 - dependence of S_τ

The supremum, infimum, and area of the set S_τ is independent of the joint variable q_1. For two manipulator positions (q_1, q_2) and ($q_1+\phi$, q_2),

$$S_\tau(q_1+\phi, q_2) = \mathbf{R}(\phi)S_\tau(q_1, q_2) \tag{91}$$

•**Proof:**

S_τ is a linear mapped set between actuator torques and the end-effector accelerations. So, property 2 is proved if the vertices of $S_\tau(q_1+\phi, q_2)$ are the simple rotation of $S_\tau(q_1, q_2)$. Components of vertex A are

$$\begin{aligned} a_{11}\tau_{1o}+a_{12}\tau_{2o} &= (\cos q_1 p_{11}-\sin q_1 p_{21})\tau_{1o}+(\cos q_1 p_{12}-\sin q_1 p_{22})\tau_{2o} \\ &= \cos q_1(p_{11}\tau_{1o}+p_{12}\tau_{2o})-\sin q_1(p_{21}\tau_{1o}+p_{22}2o) \\ a_{21}\tau_{1o}+a_{22}\tau_{2o} &= (\sin q_1 p_{11}+\cos q_1 p_{21})\tau_{1o}+(\sin q_1 p_{12}+\cos q_1 p_{22})\tau_{2o} \end{aligned} \tag{92}$$

$$= \sin q_1(p_{11}\tau_{1o}+p_{12}\tau_{2o})+\cos q_1(p_{21}\tau_{1o}+p_{22}2o) \tag{93}$$

Rewriting equations (92) and (93),

$$[a_{11}\tau_{1o}+a_{12}\tau_{2o} \;\; a_{21}\tau_{1o}+a_{22}\tau_{2o}]^T=\mathbf{R}(\phi) \;\; [p_{11}\tau_{1o}+p_{12}\tau_{2o} \;\; p_{21}\tau_{1o}+p_{22}\tau_{2o}]^T \tag{94}$$

Similarly, other vertices of $S(q_1+\phi, q_2)$ can be shown as a product of $\mathbf{R}(\phi)$ and $S_t(q_1, q_2)$. Therefore,

$$S_\tau(q_1+\phi, q_2) = \mathbf{R}(\phi)S_\tau(q_1, q_2)$$

•**Property 3**:q_2 - dependence of S_τ

The supremum, infimum, and area of the set S_τ depends only on the joint variable q_2.

We merely state this property to emphasize that the size of S_τ depends only on q_2, a property which is to be expected since, everything else being the same, two positions for which q_2 is identical [i.e. (q_1, q_2) and (q_1', q_2)] are equivalent from a kinematic and dynamic standpoint. The property follows obviously from the equations (72)-(74) which depend only on q_2.

5 Quadratic mapping

For the given (constraint) set F of allowable rates-of-change, described by (10)and represented by (the interior and the boundary of) the rectangle $E_1F_1E_2F_2$ shown in Figure 6-a, we defined the set $S_{\dot{q}}$ of all allowable $\alpha_{\dot{q}}$ in section 2 as

$$S_{\dot{q}} = \{\alpha_{\dot{q}} \mid (\exists \dot{\mathbf{q}} \in F)(\alpha_{\dot{q}} = \mathbf{B}\{\dot{\mathbf{q}}\}^2)\}$$

As in the previous section, the following questions are important

1. How is $S_{\dot{q}}$ described ?
2. What is the size of $S_{\dot{q}}$? Specifically, what is its infimum and supremum ?
3. How does $S_{\dot{q}}$ depend on the joint-variables q_1 and q_2.

5.1 Description of $S_{\dot{q}}$

If we define a vector

$$\mathbf{y} := [y_1 \;\; y_2]^T := [\dot{q}_1^{\,2} \;\; (\dot{q}_1+\dot{q}_2)^2-\dot{q}_1^{\,2}]^T \tag{95}$$

then equation (32) can be expressed as

$$\alpha_{\dot{q}} = \mathbf{B}(\mathbf{q}) \;\; \mathbf{y} \tag{96}$$

Therefore the mapping (32) can be viewed as the product of the quadratic mapping (95) from the $\dot{\mathbf{q}}$ - plane to **y** - plane followed by the linear mapping (96) from the **y** - plane to the $\alpha_{\dot{q}}$ plane.

The quadratic mapping (95) maps the constraint set F in the $\dot{\mathbf{q}}$ - plane into a set in the **y** - plane which we denote by S_y. Then the linear mapping (96) maps this set S_y into a set in the $\alpha_{\dot{q}}$ plane which is simply the set $S_{\dot{q}}$ defined in (35).

We will therefore first obtain the set S_y from the constraint set F under the quadratic mapping (95). $S_{\dot{q}}$ is then determined from S_y under the linear mapping (96).

5.1.1 The quadratic map and the description of S_y

Formally, we define S_y as follows:

$$S_y = \{\mathbf{y} \mid (\exists \dot{\mathbf{q}} \in F)(\mathbf{y} = \{\dot{\mathbf{q}}\}^2)\} \tag{97}$$

Using equation (21) we can write (95) explicitly as

$$y_1 = \dot{q}_1^{\,2} \tag{98}$$

$$y_2 = (\dot{q}_1+\dot{q}_2)^2-\dot{q}_1^{\,2} \tag{99}$$

We now have to determine the mapping of (the interior and the boundary of) the $\dot{\mathbf{q}}$ - plane rectangle $E_1F_1E_2F_2$ into the **y** - plane as determined by (98) and (99).

The notation

$$X_1 \;\rightarrow\; X \tag{100}$$

will be used to denote the fact that the point X_1 in the $\dot{\mathbf{q}}$ - plane maps into the point X in the **y** - plane, i.e. X is the image of X_1.

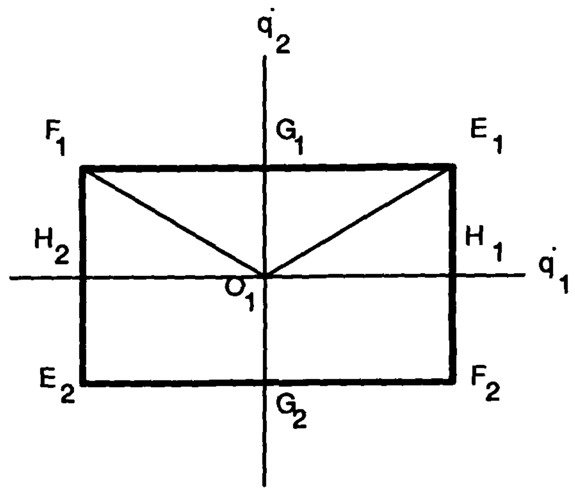

Figure 4: an available joint velocity set

From equations (98) and (99) we use that the pair of points $X_1(\dot{q}_1, \dot{q}_2)$ and $X_2(-\dot{q}_1, -\dot{q}_2)$ in the $\dot{\mathbf{q}}$ - plane both have the same image $X(\dot{q}_1^2, (\dot{q}_1+\dot{q}_2)^2-\dot{q}_1^2)$ in the y - plane, i.e.

$$X_1(\dot{q}_1, \dot{q}_2), X_2(-\dot{q}_1, -\dot{q}_2) \rightarrow X(\dot{q}_1^2, (\dot{q}_1+\dot{q}_2)^2-\dot{q}_1^2) \tag{101}$$

Consider the rectangle $E_1F_1E_2F_2$ in Figure 4. A consequence of (101) is that the quadrants $O_1H_1E_1G_1$ and $O_1H_2E_2G_2$ of $E_1F_1E_2F_2$ both map into the same region of the y - plane; Similarly the quadrants $O_1G_1F_1H_2$ and $O_1G_2F_2H_1$ (of $E_1F_1E_2F_2$) both map into the same region of the y - plane. Therefore we only need to determine the region of the y - plane into which the "upper-half" $H_1E_1F_1H_2$ (of the rectangle $E_1F_1E_2F_2$) maps. Formally $H_1E_1F_1H_2$ is described:

$$F' = \{\dot{\mathbf{q}} \mid (\exists\dot{\mathbf{q}} \in F) \ (\dot{q}_2 \geq 0)\} \tag{102}$$

The required set S_y is therefore the image of F' under the quadratic mapping (98) and (99). To determine S_y we first need to establish the following:

1. the image of the points O_1, H_1, E_1, G_1, F_1, and H_2 under the mapping (98) and (99).
2. the image of a line

$$\dot{q}_2 = m\dot{q}_1 \tag{103}$$

of slope m passing through the origin O_1.

If O, H, E, G, F denote the image of points O_1, H_1, E_1, G_1, and F_1, then from (98) and (99) we can write

$$O_1 \ (0, 0) \rightarrow O \ (0, 0) \tag{104}$$

$$H_1 \ (\dot{q}_{1o}, 0), H_2 \ (-\dot{q}_{1o}, 0) \rightarrow H \ (\dot{q}^2_{1o}, 0) \tag{105}$$

$$E_1 \; (\dot{q}_{1o}, \dot{q}_{2o}) \; \rightarrow \; E(\dot{q}_{1o}{}^2 \; (\dot{q}_{1o}+\dot{q}_{2o})^2-\dot{q}_{1o}{}^2) \tag{106}$$

$$F_1 \; (-\dot{q}_{1o}, \dot{q}_{2o}) \; \rightarrow \; F \; (\dot{q}_{1o}{}^2 \; (\dot{q}_{1o}-\dot{q}_{2o})^2-\dot{q}_{1o}{}^2) \tag{107}$$

$$G_1 \; (0, \dot{q}_{2o}) \; \rightarrow \; G \; (0, \dot{q}^2_{2o}) \tag{108}$$

Note that the points H_1 and H_2 have the same image as is to be expected from (101). Also, the origin O_1 of the $\dot{\mathbf{q}}$ - plane maps into the origin O of the **y** - plane. Using (98) and (99), the line (103) in the **y** - plane maps into the set of points

$$y_1 = \dot{q}_{1o}{}^2 \tag{109}$$

$$y_2 = (\dot{q}_1+m\dot{q}_1)^2-\dot{q}_1{}^2 = \dot{q}^2{}_1(m^2 + 2m) \tag{110}$$

Equations (109) and (110) are the parametric equations of the straight line

$$y_2 = (m^2 + 2m) \; y_1. \tag{111}$$

Therefore a line passing through the origin and of slope m in the $\dot{\mathbf{q}}$ - plane maps into a line passing through the origin and of slope ($m^2 + 2$ m) in the **y** - plane.

To obtain the image in the **y** - plane of the rectangle $H_1E_1F_1H_2$, it is convenient to divide $H_1E_1F_1H_2$ into four triangular sections $O_1H_1E_1$, $O_1E_1G_1$, $O_1G_1F_1$, and $O_1F_1H_2$ and separately determine the image set for each of these sections. The required image set is simply the union of the four image sets.

In order to determine its image set, it is convenient to think of each triangular section as composed of line segments passing through the origin. This will enable us to readily determine the interior of the image set.

Image set of $O_1H_1E_1$:

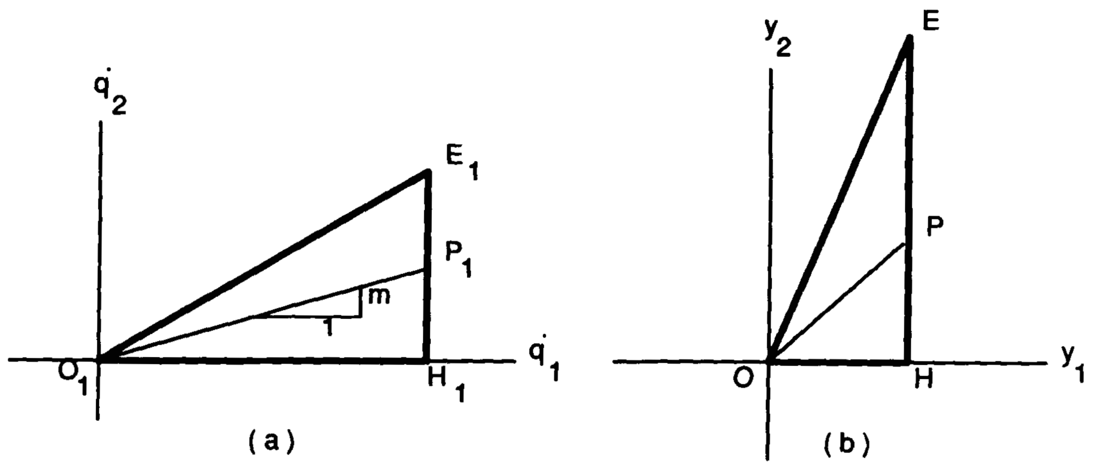

Figure 5: Image set of $O_1H_1E_1$

Let $(S_y)_1$ denote the image set of the triangle O_1 H_1E_1 (see Figure 5-a). Since a line passing through the origin O_1 in the $\dot{q}$ plane maps into a line passing through the origin of the y - plane described by equation (103), the image of the generic line segment O_1P_1 of slope m passing through O_1, shown in Figure 5-a, will be a line segment of slope $(m^2 + 2m)$ passing through the origin O in the y - plane. We now only need to determine the images of the end points O_1 (0, 0) and P_1 ($\dot{q}_{1o}$, $\dot{q}_2$) of the O_1P_1 . The image of O_1, is of course, (see equation (104).), the origin O of the y - plane. Let P denote the image of P_1. Then the line segment O_1P_1 maps into the line segment OP with one end-point at the origin O of the y - plane. All we need to do now is to determine the locus of P as P_1 moves along the line segment H_1E_1. Using equations (98) and (99), the image of the line segment H_1E_1 described by the quations

$$\dot{q}_1 = \dot{q}_{1o},\ (0 \le \dot{q}_2 \le \dot{q}_{2o}) \tag{112}$$

is the line segment HE, described by the equation

$$y_1 = \dot{q}_{1o}^{\,2},\ (0 \le y_2 \le (\dot{q}_{1o}+\dot{q}_{2o})^2-\dot{q}_{1o}^{\,2}). \tag{113}$$

Furthermore the images H and E, respectively, of points H_1 and E_1 are given by equations (105) and (106).

Therefore

1. the locus of P in the y - plane is the line segment HE,
2. the several line O_1P_1 of slope m maps into the line OP passing through the origin O whose equation is given by (111) and whose end-point P lies on the line segment and
3. any point on O_1P_1 maps into a point on OP.

Therefore the image $(S_y)_1$ of the (interior and boundary of the) $O_1H_1E_1$ is the (interior and boundary of the) triangle OHE, shown in Figure 5-b whose vertices O, H, E are given, respectively, by equations (104), (105), and (106). (O is of course the origin of the y - plane!).

Image-Set of $O_1E_1G_1$

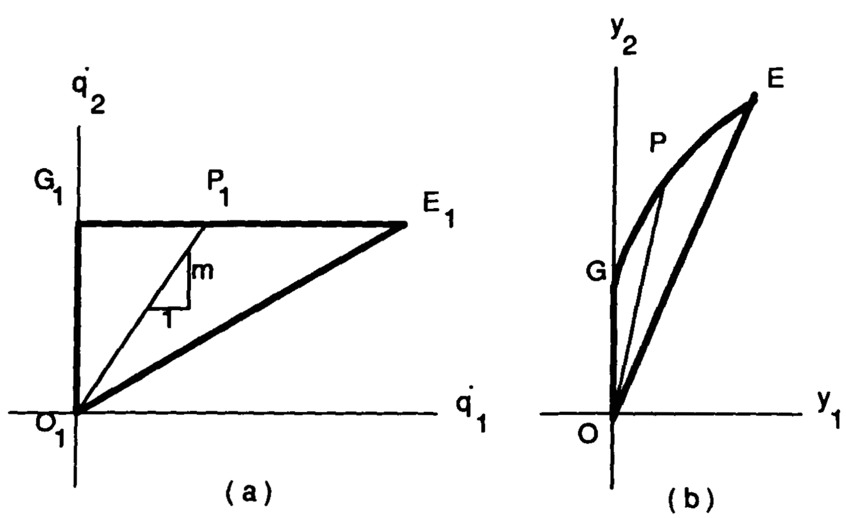

Figure 6: Image set of $O_1E_1G_1$

Let $(S_y)_2$ denote the image-set of the triangle $O_1E_1G_1$, shown in Figure 6-a. Using similar arguments as above, the generic line segment O_1P_1 of slope m, shown in Figure 6-a map into a line segment of slope ($m^2 + 2m$). If P denotes the image of P_1, then the image $(S_y)_2$ now reduces to obtaining the locus of P as P_1 moves along G_1E_1 in Figure 6-a.

From equations (98) and (99) the image of the line segment G_1E_1 described by the equation

$$\dot{q}_2 = \dot{q}_{2o}, \ (0 \le \dot{q}_1 \le \dot{q}_{1o}) \tag{114}$$

is the parabolic segment GE in the **y** - plane described by the equation,

$$(y_2 - \dot{q}_{2o}{}^2)^2 = 4\dot{q}_{2o}{}^2 y_1, \ (\dot{q}^2_{1o} \le y_2 \le (\dot{q}_{1o}+\dot{q}_{2o})^2 - \dot{q}_{1o}{}^2). \tag{115}$$

and shown in Figure 6-b.

Therefore the locus of P in the **y** - plane is the parabolic segment GE, the coordinates of whose end points G and E are given by equations (108) and (106).

The image $(S_y)_2$ of the (interior and boundary of the) triangle $O_1E_1G_1$ is the region OEG shown in Figure 6-b, whose vertices O, E and G are given, respectively, by equations (104), (106), and (108); EG is a parabolic segment described by equation (115).

The Image-Set of $O_1G_1F_1$

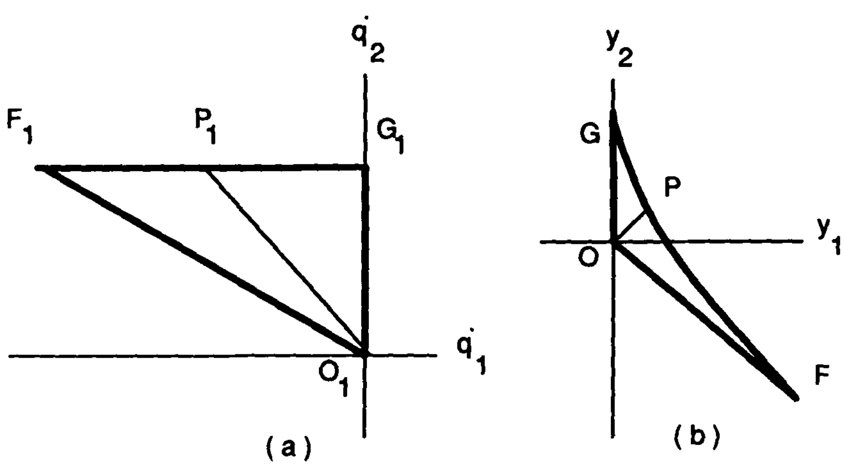

Figure 7: Image set of $O_1E_1G_1$

Let $(S_y)_3$ denote the image-set of $O_1G_1F_1$.

As before the generic-line O_1P_1 incise $O_1G_1F_1$ (see Figure 7-a.) maps into the line segment OP, where P is the image of P_1.

In this case, we have to find the locus of P as P_1 moves along G_1F_1.

Using equations (98) and (99), the image of the line segment G_1F_1, described by the equation.

$$\dot{q}_2 = \dot{q}_{2o}, \; (-\dot{q}_{1o} \le \dot{q}_1 \le 0) \tag{116}$$

is the parabolic segment GF in the **y** - plane described by the equations,

$$(y_2 - \dot{q}_{2o}^{\,2})^2 = 4\dot{q}_{2o}^{\,2}y_1, \; ((\dot{q}_{1o}-\dot{q}_{2o})^2-\dot{q}_{1o}^{\,2} \le y_2 \le \dot{q}_{1o}^{\,2}). \tag{117}$$

Therefore the locus of P in the **y** - plane is the parabolic segment GF, the coordinates of whose end-points G and F are given by equations (108) and (107).

The image $(s_y)_3$ of the (interior and boundary of the) triangular segment $O_1G_1F_1$ is the region OGF shown in Figure 7-b, whose vertices O, G, and F are given, respectively, by equations (104), (108) and (107). Note that $(s_y)_3$ is not convex.

Image-Set of $O_1F_1H_2$

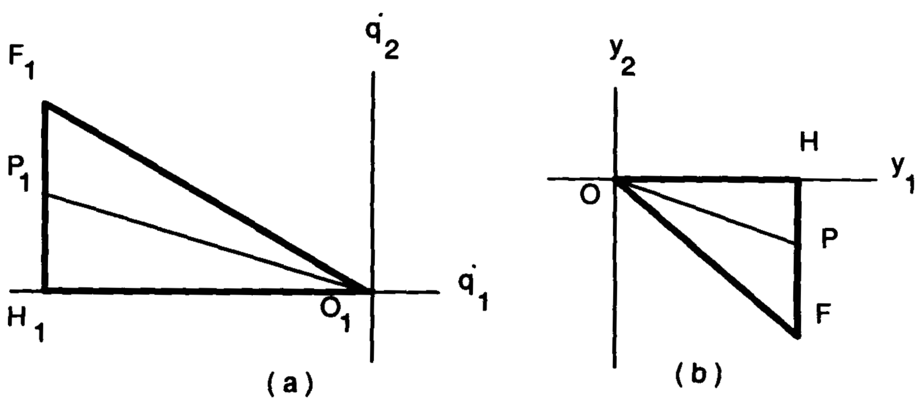

Figure 8: Image set of $O_1F_1H_2$

Let $(s_y)_4$ denote the image-set of the triangle $O_1F_1H_2$ in Figure 10-a.

Then the procedure for finding $(s_y)_4$ of the (interior and boundary of the) triangle $O_1F_1H_2$ (shown in Figure 8-a) is the triangle OFH, shown in Figure 8-b, whose vertices O, F, and H are given by equations (104), (107) and (105).

Image-Set of $H_1E_1F_1H_2$

The image-set s_y of the rectangle $H_1E_1F_1H_2$ is given by

$$s_y = \cup_{i=1,2,3,4} (s_y)_i. \tag{118}$$

s_y is shown in Figure 9.

The image-set s_y of F is the (interior and boundary of the) region OGEF, shown in Figure 11, whose vertices O, G, E and F are given, respectively, by equations (104), (108), (106) and (107). The boundaries of OGEF are the

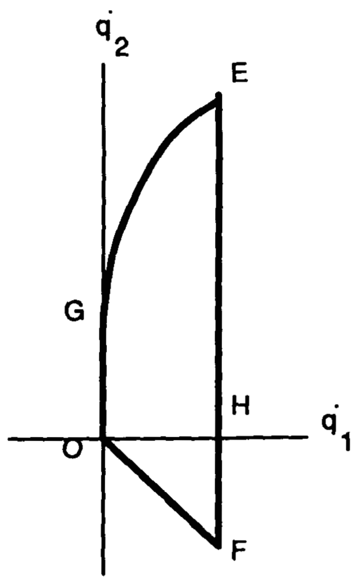

Figure 9: Image set S_u

line segments OG, OF, and FE and the parabolic segment GE whose equations are as follows:

$$OG \quad y_1 = 0, \; (0 \le y_2 \le \dot{q}^2_{2o}) \tag{119}$$

$$OF \quad y_2 = \frac{(\dot{q}_{1o}-\dot{q}_{2o})^2-\dot{q}^2_{1o}}{\dot{q}_{1o}^2} y_1, \; (0 \le y_1 \le \dot{q}_{1o}^2) \tag{120}$$

$$FE \quad y_1=\dot{q}_{1o}^2, \; ((\dot{q}_{1o}-\dot{q}_{2o})^2-\dot{q}_{1o}^2 \le y_2 \le (\dot{q}_{1o}+\dot{q}_{2o})^2-\dot{q}_{1o}^2) \tag{121}$$

$$GE \quad (y_2 - \dot{q}_{2o}^2)^2 = 4\dot{q}_{2o}^2 y_1, \; (0 \le y_1 \le \dot{q}_{1o}^2). \tag{122}$$

Thus OGEF is completely determined. The region OGEF is convex, even though $(S_y)_3$ is non-convex. This is a consequence of the fact that the non-convex boundary of $(S_y)_3$ lies in the interior of S_y.

It will be useful in section 7 to approximate the parabolic segment EG by a straight line; consequently the region OGEF (i.e., S_y is approximated by a quadrilateral). Two approximations, shown in Figure 12, which are of interest in the sequel are the following:

1. the parabolic segment EG is approximated by the straight line segment EG joining E and G; the quadrilateral OGEF which is the corresponding approximation to the region OGEF, (S_y), will be called the **inner approximation** to S_y.
2. the parabolic segment EG is approximated by the line segment EI which is tangent to the parabolic segment at E and which intersects the y_2 - axis at I. The quadrilateral OIEF which is the corresponding approximation to the region OGEF, (S_y), will be called the **outer approximation** to S_y.

The coordinate of I are given by

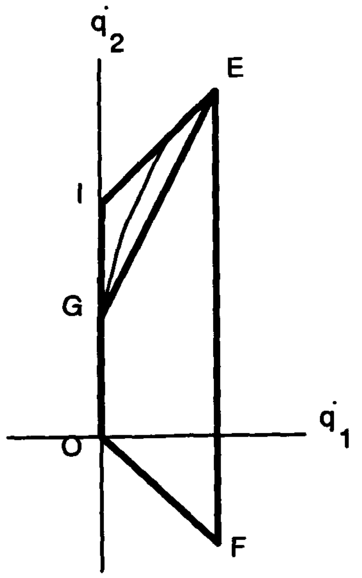

Figure 10: Approximation of Image set

$$I(\ 0,\ 2\dot{q}_{2o}{}^2\ ). \tag{123}$$

5.1.2 The linear mapping (96) and the determination of $S_{\dot{q}}$

From equation (35), the desired image set $S_{\dot{q}}$ is the image of S_y under the linear mapping (96). The matrix B(q) which characterize the linear mapping (96) is given by equation (27). Combining equations (26) and (27) and using the decompositions of the **J(q)** and **E(q)** matrices given, respectively, in equations (58) and (59), we can write **B(q)** as

$$\mathbf{B(q)} = \mathbf{R}(q_1)\ \mathbf{R}(q_2)\ \{\ -\mathbf{M}(q_2)\ \mathbf{D}^{-1}(q_2)\ \mathbf{V}(q_2)\ -\ \mathbf{N}(q_2)\ \}. \tag{124}$$

Defining the matrix $\mathbf{S}(q_2)$,

$$\mathbf{S}(q_2) = \mathbf{R}(q_2)\ \{\ -\mathbf{M}(q_2)\ \mathbf{D}^{-1}(q_2)\ \mathbf{V}(q_2)\ -\ \mathbf{N}(q_2)\ \}, \tag{125}$$

we can write **B(q)** as

$$\mathbf{B(q)} = \mathbf{R}(q_1)\ \mathbf{S}(q_2). \tag{126}$$

Therefore **B(q)** can be written as the product of a matrix **S** which is a function of q_2 only and a simple orthogonal matrix which depends on q_1 only.

The (i, j) element of the matrices $\mathbf{S}(q_2)$ and **B(q)** will be denoted, respectively, by s_{ij} and b_{ij}.

If O′, G′, E′, F′, and I′ denote, respectively, the images in the $A_{\dot{q}}$ - plane of O, G, E, F, and I, then using (96) we can write the following:

$$E(\dot q_{1o}{}^2,\ (\dot q_{1o}+\dot q_{2o})^2-\dot q_{1o}{}^2) \rightarrow E'((b_{11}-b_{12})\dot q_{1o}{}^2+b_{12}(\dot q_{1o}+\dot q_{2o})^2,\ (b_{21}-b_{22})\dot q_{1o}{}^2+b_{22}(\dot q_{1o}+\dot q_{2o})^2) \tag{127}$$

$$F(\dot q_{1o}{}^2,\ (\dot q_{1o}-\dot q_{2o})^2-\dot q_{1o}{}^2) \rightarrow F'((b_{11}-b_{12})\dot q_{1o}{}^2+b_{12}(\dot q_{1o}-\dot q_{2o})^2,\ (b_{21}-b_{22})\dot q_{1o}{}^2+b_{22}(\dot q_{1o}-\dot q_{2o})^2) \tag{128}$$

$$O \quad (0,\ 0) \quad \rightarrow \quad O' \quad (0,\ 0) \tag{129}$$

$$G \quad (0,\ \dot q_{2o}{}^2) \quad \rightarrow \quad G' \quad (b_{12}\dot q_{2o}{}^2,\ b_{22}\dot q_{2o}{}^2) \tag{130}$$

The images E′F′, O′E′, O′F′, and O′G′ of the linesegments EF, OE, OF, and OG are line segments described by the following equations:

$$E'F' \cdot \frac{1}{b_{12}}\alpha_{1\dot q} - \frac{1}{b_{22}}\alpha_{2\dot q} + \left(-\frac{b_{11}}{b_{12}}+\frac{b_{21}}{b_{22}}\right)\dot q_{1o}{}^2 = 0 \tag{131}$$

$$O'E' \cdot \frac{\alpha_{1\dot q}}{(b_{11}-b_{12})\dot q_{1o}{}^2+b_{12}(\dot q_{1o}+\dot q_{2o})^2} = \frac{\alpha_{2\dot q}}{(b_{21}-b_{22})\dot q_{1o}{}^2+b_{22}(\dot q_{1o}+\dot q_{2o})^2} \tag{132}$$

$$O'F' \cdot \frac{\alpha_{1\dot q}}{(b_{11}-b_{12})\dot q_{1o}{}^2+b_{12}(\dot q_{1o}-\dot q_{2o})^2} = \frac{\alpha_{2\dot q}}{(b_{21}-b_{22})\dot q_{1o}{}^2+b_{22}(\dot q_{1o}-\dot q_{2o})^2} \tag{133}$$

$$O'G' \cdot \frac{\alpha_{1\dot q}}{b_{12}} = \frac{\alpha_{2\dot q}}{b_{22}}. \tag{134}$$

The parabolic segment EG maps into a parabolic segment G′E′.

We can therefore write: The image set $S_{\dot q}$ of the set F is the (interior and boundary) of the region E′F′O′G′, shown in Figure 11, whose vertices E′, F′, O′, and G′ are given, respectively, by equations (127), (128), (130), and (129). The segments E′F′, O′E′, O′F′, and O′G′ are given, respectively, by equations (131), (132), (133), and (134). Thus $S_{\dot q}$ is completely determined.

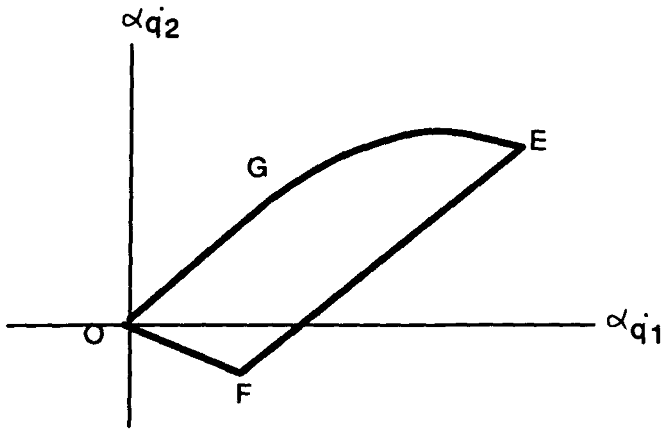

Figure 11: Image set $S_{\dot q}$

5.2 Properties of $S_{\dot{q}}$

In this section, we derive expressions for the supremum and infimum of $S_{\dot{q}}$.

5.2.1 Furthest vertex of $S_{\dot{q}}$

We first show that the furthest vertex of $S_{\dot{q}}$ is E′. Inspection of equations (119) through (122) shows that the furthest vertex of the set S_y is E. Since the set $S_{\dot{q}}$ is the image of S_y under a linear mapping, the furthest vertex of $S_{\dot{q}}$ is the image of the furthest vertex, E, of S_y. Therefore E′ is the furthest vertex of $S_{\dot{q}}$.

5.2.2 Supremum of $S_{\dot{q}}$

The supremum of $S_{\dot{q}}$ is the distance of the furthest vertex E′ from the origin. Using equation (127), we obtain

$$\sup(S_{\dot{q}}) =$$

$$[(s_{11}-s_{12})^2+(s_{21}-s_{22})^2]\dot{q}_{1o}{}^4+[s_{12}{}^2+s_{22}{}^2](\dot{q}_{1o}+\dot{q}_{2o})^4+2[s_{12}(s_{11}-s_{12})+s_{22}(s_{21}-s_{22})]\dot{q}_{1o}{}^2(\dot{q}_{1o}+\dot{q}_{2o})^2. \quad (135)$$

5.2.3 Infimum of $S_{\dot{q}}$

Since the origin O′ (0, 0) is one of the vertices of $S_{\dot{q}}$, the infimum of $S_{\dot{q}}$ is zero:

$$\inf(S_{\dot{q}}) = 0 \quad (136)$$

6 The state acceleration set, S_u

We defined the acceleration set, S_u, at a specified point **u** in the state space as follows:

Definition of $S_u(q,\dot{q})$: For a given set T of allowable actuator torques described by

$$T = \{\tau \mid |\tau_i| \le \tau_{io},\ i=1,2\},$$

the acceleration set S_u at a point $\mathbf{u} = [\,\mathbf{q}, \dot{\mathbf{q}}\,]^T$ in the state - space is given by

$$S_u(q,\dot{q}) = \{\alpha_u \mid (u=(q,\dot{q})),\ (\exists\tau \in T)(\tau_u = A(q)\tau + k(u))\}$$

Thus S_u is the image of the set T under the mapping (38).

6.1 Determination of S_u

Inspection of equation (31) and (38) reveals that

$$\alpha_u(q,\dot{q},\tau) = \alpha_\tau(q,\tau) + k(q,\dot{q}) \tag{137}$$

where $\alpha_\tau(q\,,\tau) \in S_\tau(q,\tau)$ and $\alpha_u(q,\dot{q},\tau) \in S_u(q,\dot{q},\tau)$.

From (137), we see that

$$\alpha_u(q,\ \dot{q}=0,\ \tau) = \alpha_\tau(q,\ \tau). \tag{138}$$

Defining,

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{139}$$

and

$$\bar{\alpha}_u = \alpha_u(q,\ \dot{q}=0,\ \tau) \tag{140}$$

then

$$\bar{\alpha}_u = I\ \alpha_\tau(q,\ \tau) \tag{141}$$

and

$$\alpha_u = \bar{\alpha}_u + k(q,\dot{q}). \tag{142}$$

If we define a set $\bar{S}_u$,

$$\bar{S}_u = \{\bar{\alpha}_u \mid (\exists\alpha_\tau \in S_\tau)(\bar{\alpha}_u = I\ \alpha_\tau)\}, \tag{143}$$

then $\bar{S}_u$ is the image in the A_u - plane of the set $\bar{S}_\tau$ in the A_τ - plane under the identity mapping (139). From (142) the desired acceleration set S_u at a specified point $\mathbf{u} = [\,\mathbf{q}, \dot{\mathbf{q}}\,]^T$ in the state space is the set obtained by translating $\bar{S}_u$ by the (constant) displacement vector $\mathbf{k}(\,\mathbf{q}, \dot{\mathbf{q}}\,)$. This process of generating S_u is shown in Figure 12-a, b, and c.

We can write S_u in the following equivalent form:

$$S_u = \{\alpha_u \mid (\exists\alpha_\tau \in S_\tau)\ (\alpha = \alpha_\tau + k)\} \tag{144}$$

Since S_τ is a parallelogram $A'B'C'D'$, $\bar{S}_u$ and S_u are also parallelograms congruent to S_τ but lying in the A_u - plane. The centroid of the set S_u has coordinates (α_{u1}, α_{u2}) as shown in Figure 12-c. Loosely speaking, we can say that S_u is obtained by translating S_τ by (α_{u1}, α_{u2}) from the origin.

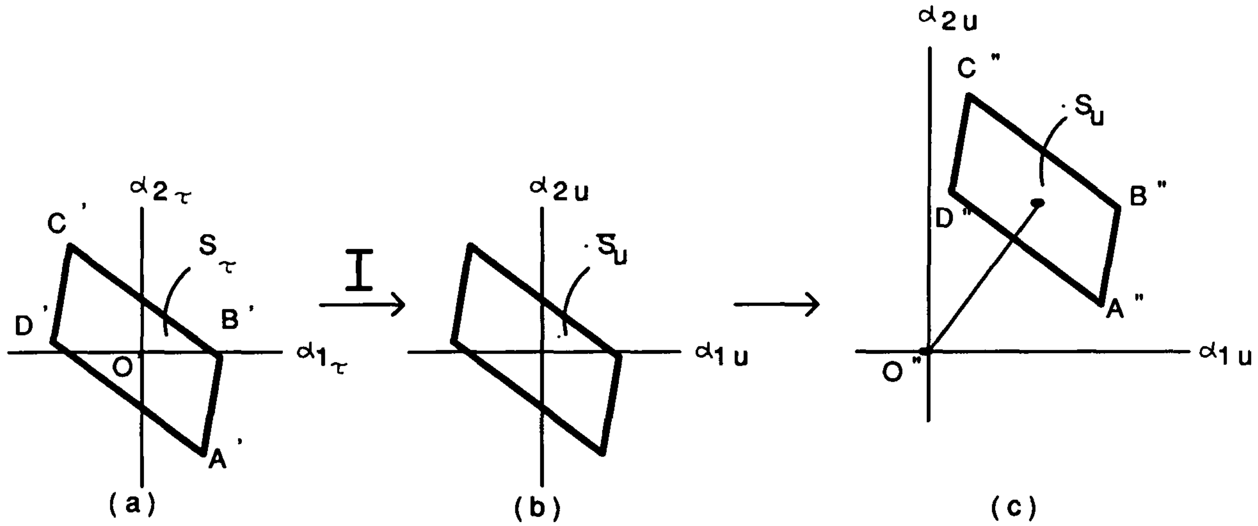

Figure 12: A state acceleration set

If A", B", C", and D" denote, respectively, the images in the A_u - plane of points A′, B′, C′, and D′ in the A_τ - plane (Figure 3-a), then from equations (137) and (64) through (67), we obtain

$$A'(a_{11}\tau_{1o}+a_{12}\tau_{2o}, a_{21}\tau_{1o}+a_{22}\tau_{2o}) \rightarrow A''(k_1+a_{11}\tau_{1o}+a_{12}\tau_{2o},\ k_2+a_{21}\tau_{1o}+a_{22}\tau_{2o}) \quad (145)$$

$$B'(a_{11}\tau_{1o}-a_{12}\tau_{2o}, a_{21}\tau_{1o}-a_{22}\tau_{2o}) \rightarrow B''(k_1+a_{11}\tau_{1o}-a_{12}\tau_{2o},\ k_2+a_{21}\tau_{1o}-a_{22}\tau_{2o}) \quad (146)$$

$$C'(-a_{11}\tau_{1o}-a_{12}\tau_{2o}, -a_{21}\tau_{1o}-a_{22}\tau_{2o}) \rightarrow C''(k_1-a_{11}\tau_{1o}-a_{12}\tau_{2o},\ k_2-a_{21}\tau_{1o}-a_{22}\tau_{2o}) \quad (147)$$

$$D'(-a_{11}\tau_{1o}+a_{12}\tau_{2o}, -a_{21}\tau_{1o}+a_{22}\tau_{2o}) \rightarrow D''(k_1-a_{11}\tau_{1o}+a_{12}\tau_{2o},\ k_2-a_{21}\tau_{1o}+a_{22}\tau_{2o}). \quad (148)$$

S_u is the (interior and boundary) of the parallelogram A"B"C"D". The sides A"B", B"C", C"D", and D"A" of the parallelogram are obtained by (137) and equations (68) through (71),

$$A''B''\cdot\ \frac{1}{a_{12}}a_{1\tau} - \frac{1}{a_{22}}a_{2\tau} + \left(-\frac{a_{11}}{a_{12}}+\frac{a_{21}}{a_{22}}\right)\tau_{1o} + \left(-\frac{k_1}{a_{12}}+\frac{k_2}{a_{22}}\right) = 0 \quad (149)$$

$$B''C''\cdot\ \frac{1}{a_{11}}a_{1\tau} - \frac{1}{a_{21}}a_{2\tau} + \left(-\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)\tau_{2o} + \left(-\frac{k_1}{a_{11}}+\frac{k_2}{a_{21}}\right) = 0 \quad (150)$$

$$C''D''\cdot\ \frac{1}{a_{12}}a_{1\tau} - \frac{1}{a_{22}}a_{2\tau} - \left(-\frac{a_{11}}{a_{12}}+\frac{a_{21}}{a_{22}}\right)\tau_{1o} + \left(-\frac{k_1}{a_{12}}+\frac{k_2}{a_{22}}\right) = 0 \quad (151)$$

$$D''A''\cdot\ \frac{1}{a_{11}}a_{1\tau} - \frac{1}{a_{21}}a_{2\tau} - \left(-\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)\tau_{2o} + \left(-\frac{k_1}{a_{11}}+\frac{k_2}{a_{21}}\right) = 0. \quad (152)$$

6.2 Supremum of S_u

The supremum of S_u is a measure of the largest acceleration available (in some direction) at a specified point in the state-space. In a similar manner to that of S_τ, the supremum of S_u is obtained as the distance of the furthest vertex of the parallelogram A"B"C"D" from the origin O" of the A_u plane.

If l(O"A"), l(O"B"), l(O"C"), and l(O"D") denote, respectively, the distances of the vertices A", B", C", and

D" from the origin O", then from (145) through (148) we obtain

$$l(O''A'') = \sqrt{(k_1+a_{11}\tau_{1o}+a_{12}\tau_{2o})^2+(k_2+a_{21}\tau_{1o}+a_{22}\tau_{2o})^2} \quad (153)$$

$$l(O''B'') = \sqrt{(k_1+a_{11}\tau_{1o}-a_{12}\tau_{2o})^2+(k_2+a_{21}\tau_{1o}-a_{22}\tau_{2o})^2} \quad (154)$$

$$l(O''C'') = \sqrt{(k_1-a_{11}\tau_{1o}-a_{12}\tau_{2o})^2+(k_2-a_{21}\tau_{1o}-a_{22}\tau_{2o})^2} \quad (155)$$

$$l(O''D'') = \sqrt{(k_1-a_{11}\tau_{1o}+a_{12}\tau_{2o})^2+(k_2-a_{21}\tau_{1o}+a_{22}\tau_{2o})^2} \quad (156)$$

The supremum of S_u is given by

$$\sup(S_u) = \max[\ l(O''A''),\ l(O''B''),\ l(O''C''),\ l(O''D'')\ ] \quad (157)$$

6.3 Infimum of S_u

The infimum is the maximum isotropic acceleration for a certain manipulator position in the workspace.

To obtain the infimum we must consider three cases

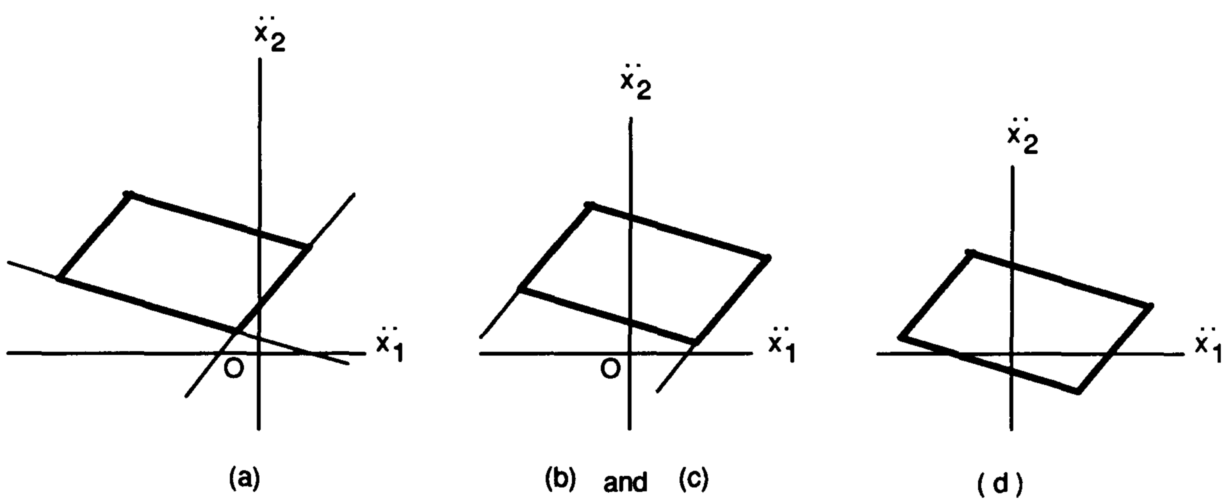

Figure 13: the relative location of a parallelogram to the origin

1. The origin O" lies outside the parallelogram A"B"C"D" and O" does not lie between either pair of parallel lines (Figure 13-a) comprising the sides of the parallelogram.
2. The origin O" lies outside the parallelogram A"B"C"D" and O" lies between A"B" and C"D" (Figure 13-b).
3. The origin O" lies outside the parallelogram A"B"C"D" and O" lies between B"C" and D"A" (Figure 13-c).
4. The origin O" lies inside the parallelogram A"B"C"D" (Figure 13-d).

Using well-known results from analytic geometry, the condition for O" to lie between the parallel lines A"B" and C"D" is obtained from (149) and (151) as

condition 1:

$$(\frac{k_1}{a_{12}}-\frac{k_2}{a_{22}})^2 \le (\frac{a_{11}}{a_{12}}-\frac{a_{21}}{a_{22}})^2\tau_{1o}^{\,2}; \qquad (158)$$

the condition for O" to lie between the parallel lines B"C" and D"A" is obtained from (150) and (152) as

condition 2:

$$(\frac{k_1}{a_{11}}-\frac{k_2}{a_{21}})^2 \le (\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}})^2\tau_{2o}^{\,2} \qquad (159)$$

Using the above two conditions, the three cases can easily be identified from the following rules:

- case 1: both conditions 1 and 2 are false.
- case 2: condition 1 is false and condition 2 is true.
- case 3: condition 1 is true and condition 2 is false.
- case 4: both conditions 1 and 2 are true.

The infimum for the three cases is obtained as follows:

case 1:(Figure 13-a)

In this case, the infimum is the distance of the closest vertex of A"B"C"D" from the origin O". Therefore

$$\inf(S_u) = \min[\ l(O''A''),\ l(O''B''),\ l(O''C''),\ l(O''D'')\ ] \qquad (160)$$

case 2:(Figure 13-b)

In this case the infimum is the distance from the origin to the nearest side, which is either A"B" or C"D".

let d(A"B") and d(C"D") be, respectively, the distances from O" to sides A"B" and C"D".

$$d(A''B''),d(C''D'') = \frac{|(a_{11}a_{22}-a_{12}a_{21})\tau_{1o}\pm(a_{22}k_1-a_{12}k_2)|}{\sqrt{a_{12}^2+a_{22}^2}} \qquad (161)$$

In a manner similar to obtaining the infimum of S_τ, the infimum of S_u is obtained from equations (161) as

$$\inf(S_u) = \min[\ d(A''B''),\ d(C''D'')\ ] \qquad (162)$$

case 3:(Figure 13-c)

The nearest side is either B"C" or D"A". let d(B"C") and d(D"A") be, respectively, the distances from O" to sides B"C" and D"A".

$$d(B''C''),d(D''A'') = \frac{|(a_{12}a_{21}-a_{11}a_{22})\tau_{2o}\pm(a_{21}k_1-a_{11}k_2)|}{\sqrt{a_{11}^2+a_{21}^2}} \qquad (163)$$

The infimum of S_u is obtained from equations (163) as

$$\inf(S_u) = \min[\ d(B''C''),\ d(D''A'')\ ] \qquad (164)$$

case 4: (Figure 13-d) The infimum is the distance from the origin to the nearest side which could be either A"B", B"C", C"D", or D"A". These distances were computed for cases 2 and 3 above. Therefore,

$$\inf(S_u) = \min[\ d(A''B''),\ d(B''C''),\ d(C''D'''',\ d(D''A'')\ ] \qquad (165)$$

To summarize the results of this section we can state the following lemma.

Lemma: The acceleration set s_u at a point **u** in the state-space of the manipulator is a parallelogram with centroid located at the point (k_1, k_2) defined by equations (37); the supremum of s_u is given by (157) and the infimum of s_u is given by one of equations (160), (162), (164), and (165). The supremum and infimum of s_u is independent of the joint angle q_1.

7 Local acceleration sets

At a given position $\mathbf{q} = [\, q_1, q_2 \,]^T$ in the workspace of the manipulator, we could define two sets in section 2

$$(s_L)_1 = \cup_{\dot{q} \in F} s_u(q,\dot{q})$$

$$(s_L)_2 = \cap_{\dot{q} \in F} s_u(q,\dot{q})$$

The supremum of $(s_L)_1$ will be give us the magnitude of the maximum acceleration (in some direction) of the reference point P at a given position (q_1, q_2) of the manipulator.

The infimum of $(s_L)_2$ will give us the magnitude of the maximum acceleration of the reference point P available in all direction at a given position of the manipulator. The infimum of $(s_L)_2$ is called the isotropic acceleration in Khatib [1] and the local acceleration radius Kim [5].

7.1 Determination of $(S_L)_1$

The generic member s_u of the set $(s_L)_1$ was described in section 6 and is shown in Figure 9. As $\dot{q}$ is varied, $s_{\dot{q}}$ is a parallelogram which moves parallel to itself. The locus of the centroid, (k_1, k_2), of the parallelogram as $\dot{q}$ is varied is simply the boundary $O'G'E'F'$ of the set $s_{\dot{q}}$ shown in Figure 11. Therefore we can describe $(s_L)_1$ as follows: The local acceleration set $(s_L)_1$ is the region swept out by the parallelogram s_u as its vertex moves along the boundary $O'G'E'F'$. This is shown in Figure 14.

7.2 Supremum of $(S_L)_1$

The supremum of $(s_L)_1$ is simply the distance of the origin from the furthest point of $(s_L)_1$.

To determine the furthest point of $(s_L)_1$, all we need to do is to determine

1. the furthest vertex of $O'G'E'F'$,
2. the parallelogram at the furthest vertex, and
3. the furthest vertex of this parallelogram

In section 5, we showed that E' is the furthest vertex of $O'G'E'F'$. The distances d($O'A'_{E'}$), d($O'B'_{E'}$), d($O'C'_{E'}$), d($O'D'_{E'}$) of the vertices of the parallelogram with centroid E' are given by (153) through (156). The supremum of $(s_L)_1$ is now readily obtained as

$$\sup(s_L)_1 = \max[\; d(O'A'_{E'}),\; d(O'B'_{E'}),\; d(O'C'_{E'}),\; d(O'D'_{E'}) \;]. \tag{166}$$

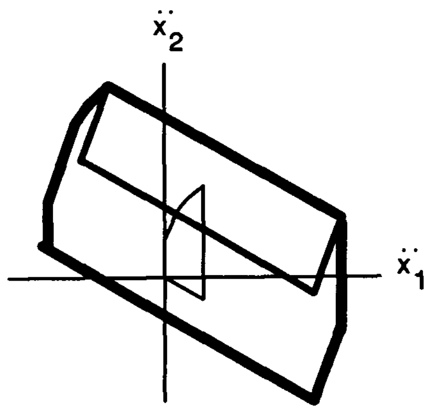

Figure 14: Determination of $(S_L)_1$

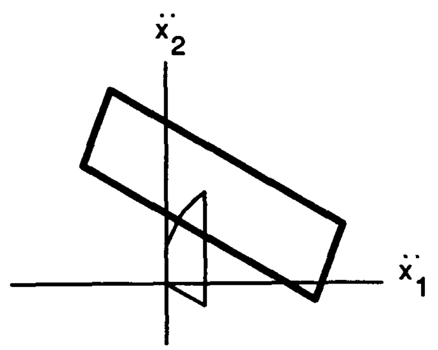

Figure 15: A local supremum

7.3 Determination of $(S_L)_2$

Using reasoning similar to that in the above section we can describe $(S_L)_2$ as follows: The local acceleration set $(S_L)_2$ is the largest region common to all the parallelograms generated by moving the generic parallelogram S_u along the the boundary $O'G'E'F'$.

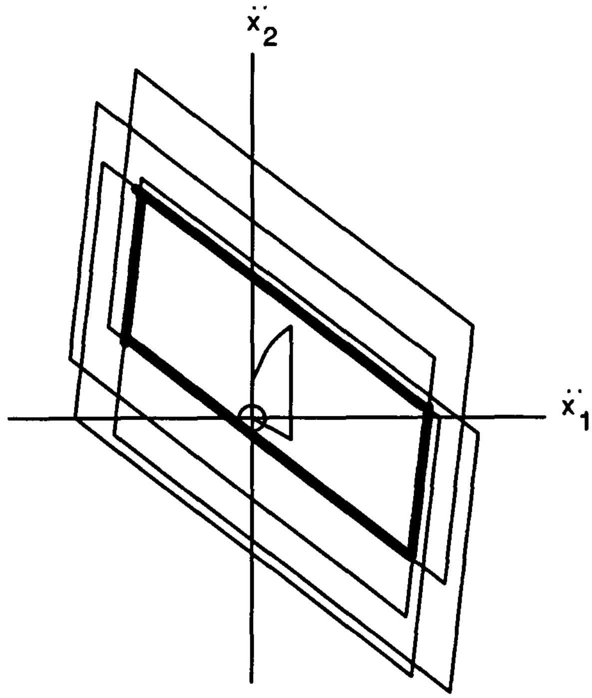

Figure 16: $(S_L)_2$ and a local infimum

7.4 Infimum of $(S_L)_2$

The infimum of $(S_L)_2$ is the maximum distance to the origin from the boundary of $(S_L)_2$.

To determine the infimum of $(S_L)_2$, the inner and outer approximation of the set $S_{\dot{q}}$ in section 5 (and consequently the boundary $O'G'E'F'$) are useful.

The problem of determining the infimum corresponding to an approximating quadrilateral reduces to examining the parallelograms with centroids at the vertices of the approximating quadrilateral since these represent the extreme parallelograms.

The procedure for finding the infimum, r, corresponding to an approximate quadrilateral (inner or outer) is as follows:

1. Construct the parallelogram at each of the four vertices O', G', E', F' or O', I', E', F' of the quadrilateral. Let P_i, (i = 1, 2, 3, 4) denote these four parallelograms.
2. Check if each parallelogram P_i satisfies the two conditions (158) and (159). If all the parallelograms satisfy these conditions, then an infimum exists.
3. For each parallelogram, P_i, determine the minimum distance, d_i, from the origin to the four sides of P_i.
4. Then the infimum, r, of $(S_L)_2$ for the approximation is given by

$$r = \inf(S_L)_2 = \min(d_i, i=1,2,3,4) \tag{167}$$

Let r_1 and r_2 denote, respectively, the infimum corresponding to the inner approximation OGEF and OIEF.

We now need to distinguish 3 cases.

- **case 1:** $r_2 = \min\ (d_i, i = 1, 2, 3, 4)$ was obtained from the parallelogram with vertex I. In this case r_1 and

r_2 are different and

$$r_1 < \inf(S_L)_2 < r_2 \tag{168}$$

- **case 2:** $r_1 = \min(d_i, i = 1, 2, 3, 4)$ was obtained from the parallelogram with vertex G. In this case r_1 and r_2 are different and

$$r_1 < \inf(S_L)_2 < r_2 \tag{169}$$

- **case 3:** r_1 is not obtained from the parallelogram with vertex G and r_2 is not obtained from the parallelogram with vertex I. In this case r_1 and r_2 are both obtained from one of the other three vertices and therefore $r_1 = r_2$ and

$$\inf(S_L)_2 = r_1 = r_2 \tag{170}$$

Therefore we either obtain the inf $(S_L)_2$ exactly as in equation (170) or with tight bounds as in equations (168) or (169).

8 Example

In this section, we will illustrate the manipulator dynamic properties obtained in section 4 through 7 using a two degree-of-freedom manipulator. First, we show the state accelerations at the vertices of joint velocity contribution quadrilateral in section 6. The contribution, α_τ, of actuator torques in section 4 is the local acceleration set with the zero joint velocity vector. Then, using the state supremum and infimum in section 6, we illustrate the manipulator state performance. Finally, the local supremum and infimum in section 7 is calculated.

To provide an experimental test-bed, we have built a two degree-of-freedom planar revolute-jointed manipulator, shown schematically in Figure 1. The design variables of the manipulator consist of

l_1= 0.303 m	l_2= 0.254 m
a_1= 0.196 m	a_2= 0.0941 m
m_1= 2.26 kg	m_2= 0.177 kg
I_1= 0.129 kg m^2	I_2= 2.77×10^{-3} kg m^2

The actuator torque set is

$$T = \{ \tau \mid |\tau_i| \le 30.\ Nm,\ i=1,2\},$$

the joint-velocity set is

$$V = \{\dot{q} \mid |\dot{q}_i| \le 1.0\ rad/sec,\ i=1,2\},$$

and the workspace is

$$W = \{q \mid 1. \le q_2 \le \pi\ rad\}.$$

We choose the manipulator position as $\mathbf{q} = [\ 0,\ \pi/2\ ]^T$. Our first step is to calculate the elements of matrices **A**, **B** for the manipulator position as follows;

a_{11}= 0.000	a_{12}= -58.666	a_{21}=1.308	a_{22}=-1.308
b_{11}= -0.007	b_{12}= -0.000	b_{21}= -0.247	b_{22}=-0.247

Using equation (157), the state supremum at point O′ of section 6 are calculated as 1761.73 m/sec^2. Since the parallelogram of section 4 is the state acceleration at point O′, the supremum of the contribution, α_τ, of actuator torques in section 4 is also 1761.73 m/sec^2.

To obtain the state infimum, two conditions (158) and (159) are tested. Two conditions for our manipulator are both positive, and equation (165) for case 4 is used to calculate the state infimum for $[0,\ 1.57,\ 0,\ 0\ ]^T$. The infimum for $[\ 0,\ 1.57,\ 0,\ 0\ ]^T$ is 39.22 m/sec^2.

The local supremum is the supremum of the state acceleration located at point O′ in Figure 17. From equation (166), the supremum of position $[\ 0,\ 1.57\ ]^T$ is obtained as 1761.78 m/sec^2. To calculate the local infimum, two sets of state accelerations should be considered as in section 7. The infimums for the state acceleration sets at point O′, F′, E′, G′, I′, of section 6 are as follows:

point O	point F	point E	point G	point I
1761.73 m/sec^2	1761.73 m/sec^2	1761.78 m/sec^2	1761.74 m/sec^2	1761.74 m/sec^2

Among these infimums, the minimum is the infimum as in equation (170). So, the local infimum for $[\ 0,\ 1.57\ ]^T$ is 38.23 m/sec^2.

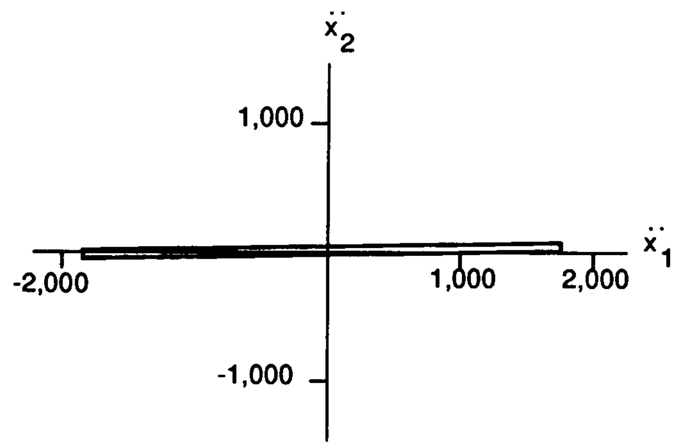

Figure 17: The acceleration set for a dynamic state O

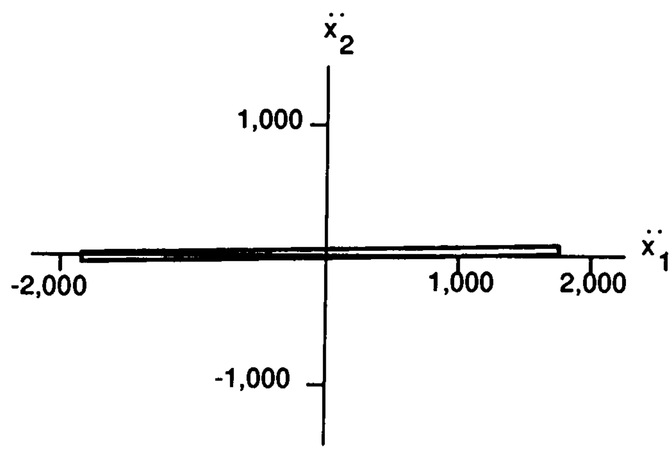

Figure 18: The local acceleration set

I. Equations of motion for a two-degree-of-freedom planar manipulator

1. Jacobian matrix

The Jacobian matrix **J** has the following components:

$$j_{11}=-l_1\sin q_1-l_2\sin(q_1+q_2)$$

$$j_{12}=-l_2\sin(q_1+q_2)$$

$$j_{21}=l_1\cos q_1+l_2\cos(q_1+q_2)$$

$$j_{22}=l_2\cos(q_1+q_2)$$

When this relationship is differentiated with respect to the time, we obtain the following equation.

$$\ddot{x}=\mathbf{J}\ddot{q}+\dot{\mathbf{J}}\dot{q}=\mathbf{J}\ddot{q}-\mathbf{E}\{\dot{q}\}^2$$

where **E** is a (2×2) matrix which has the following elements:

$$e_{11}=l_1\cos q_1+l_2\cos(q_1+q_2)$$

$$e_{12}=l_2\cos(q_1+q_2)$$

$$e_{21}=l_1\sin q_1+l_2\sin(q_1+q_2)$$

$$e_{22}=l_2\sin(q_1+q_2)$$

2. Dynamic equation

The dynamics of a two-degree-of-freedom planar manipulator is described by the following equation:

$$\mathbf{D}\ddot{q}+\mathbf{V}\{\dot{q}\}^2+p=\tau$$

D is a (2×2) matrix and the components are as follows:

$$d_{11}=I_1+m_1a_1^2+I_2+m_2(a_2^2+2a_2l_1\cos q_2+l_1^2)$$

$$d_{12}=I_2+m_2(a_2^2+a_2l_1\cos q_2)$$

$$d_{12}=d_{21}$$

$$d_{22}=I_2+m_2a_2^2.$$

V is also a (2×2) matrix and has a following components:

$$v_{11}=0$$

$$v_{12}=-v$$

$$v_{21}=v$$

$$v_{22}=0.$$

where

$$v=m_2a_2l_1\sin q_2.$$

$p=[p_1\ p_2]^T$ is a vector with the rank 2.

$$p_1=m_1ga_1\sin q_1+m_2g[l_1\sin q_1+a_2\sin(q_1+q_2)]$$

$$p_2=m_2ga_2\sin(q_1+q_2)$$

where g is a gravitational constant.

3. Acceleration equation

The expression of the acceleration of the end-effector consists of three components as follows:

$$\ddot{x} = \mathbf{A}\,\tau + \mathbf{B}\{\dot{q}\}^2 + c$$

where

$$\mathbf{A} = \mathbf{J}\,\mathbf{D}^{-1}$$

$$\mathbf{B} = -\mathbf{A}\,\mathbf{V} - \mathbf{E}$$

$$c = -\mathbf{A}\,p$$

A is a (2×2) matrix and has the following components:

$$a_{11}=\Delta(j_{11}d_{22}-j_{12}d_{12})$$

$$a_{12}=\Delta(-j_{11}d_{12}+j_{12}d_{11})$$

$$a_{21}=\Delta(j_{21}d_{22}-j_{22}d_{12})$$

$$a_{22}=\Delta(-j_{21}d_{12}+j_{22}d_{11})$$

where

$$\Delta=[d_{11}d_{22}-d_{12}{}^{2}]^{-1}$$

B is also a (2×2) matrix and the elements are as follows:

$$b_{11}=-va_{12}-e_{11}$$

$$b_{12}=va_{11}-e_{12}$$

$$b_{21}=-va_{22}-e_{21}$$

$$b_{22}=va_{21}-e_{22}$$

$c = [c_1\ c_2]^T$ is a vector with the rank 2.

$$c_1=a_{11}p_1+a_{12}p_2$$

$$c_2=a_{21}p_1+a_{22}p_2$$

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