CARNEGIE-MELLON UNIVERSITY

Robotics Institute
Vision Laboratory

Kinematics of DDArm II

Pradeep K. Khosla and Takeo Kanade

6 January 1986

Robotics Institute Document RVL017

Location of source file: /usrr0/pkk/project/reports/kin.mss

1. Introduction

The kinematics **of** a robot are completely described by the Forward solution. the Reverse solution, the Jacobian and the inverse Jacobian. This paper describes the derivation of the kinematics of the CMU-DDArm II robot.

The Forward solution is a **4** by **4** matrix that specifies **the** position and orientation of the end effector with respect to the base frame. This solution **is** denoted by the T_6 matrix and is a function of the six joint variables only [3]. We normally know where we want to move the manipulator in terms of the T_6 matrix and it is desired to obtain the joint coordinates in order to make the **move. The** transformation relating the T_6 matrix to the values of the joint coordinates is called the Reverse solution. The Forward arid the Reverse solutions are derived in Sections 3 arid **4** respectively.

Differential relationships are important to a manipulator in many ways. The transformation relating the differential changes in the joint coordinates **to** the differential changes in the world coordinates is called the Jacobian and is specified by a 6 by 6 matrix. The Jacobian of the DDArm **II** is derived in Section 5.

2. Assignment of Coordinate Frames

The coordinate frames for each link on DDArm II have been assigned according to the Denavit and Hartenberg convention [1] and are depicted in Figure 1. The link parameters of the **arm** are shown in Table 1. These **parameters arc** used as **an** input to the ARM program [2] and the **A** matrices and the Forward solution generated

3. The Forward solution

The relationship between successive **frames** $n-1$ and n (assigned according to the Denavit and Hartenberg convention) can be established by the following relationship:

- rotate about z_{n-1}, an angle, θ;
- translate along z_{n-1}, a distance, d_n;
- translate along rotated x_{n-1}, a length a_n;
- rotate about x_n, the twist angle α_n;

The product of the above four homogeneous transformations relates the coordinate frame of link n to the coordinate frame of link $n-1$ and is called the **A** matrix. **This** is represented **as,**

$$A_n = \mathrm{Rot}(z,\theta)\ \mathrm{Trans}(0,0,d)\ \mathrm{Trans}(a,0,0)\ \mathrm{Rot}(x,\alpha)$$

The matrices **A,** through **A,** arc computed using the **link** parameters listed **in Table 1. These are:**

$$A_1=\begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$A_2=\begin{bmatrix} C_2 & 0 & S_2 & a_2C_2 \\ S_2 & 0 & -C_2 & a_2S_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$A3=\begin{bmatrix} C_3 & 0 & -S_3 & 0 \\ S_3 & 0 & C_3 & 0 \\ 0 & -1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$A_4=\begin{bmatrix} C_4 & 0 & S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$A_5=\begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$A_6=\begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying (1) through (6), we obtain the description of the end effector of the manipulator with respect to the base frame.

$$T_6 = A_1A_2A_3A_4A_5A_6 \tag{7}$$

The column vectors of the T_6 matrix are given as:

$$T_6[*][1] = \begin{bmatrix} C_4S_{12}S_6 + C_{12}C_3S_4S_6 - C_{12}C_6S_3S_5 - C_5C_6S_{12}S_4 + C_{12}C_3C_4C_5C_6 \\ -C_{12}C_4S_6 + C_3S_{12}S_4S_6 - C_6S_{12}S_3S_5 + C_{12}C_5C_6S_4 + C_3C_4C_5C_6S_{12} \\ S_3S_4S_6 + C_3C_6S_5 + C_4C_5C_6S_3 \\ 0 \end{bmatrix} \tag{8}$$

$$T_6[*][2] = \begin{bmatrix} C_4S_{12}C_6 + C_{12}C_3S_4C_6 + C_{12}S_6S_3S_5 + C_5S_6S_{12}S_4 - C_{12}C_3C_4C_5S_6 \\ -C_{12}C_4C_6 + S_3S_{12}S_5S_6 + C_6S_{12}C_3S_4 - C_{12}C_5S_6S_4 - C_3C_4C_5S_6S_{12} \\ S_3S_4C_6 - C_3S_6S_5 - C_4C_5S_6S_3 \\ 0 \end{bmatrix} \tag{9}$$

$$T_6[*][3] = \begin{bmatrix} C_{12}C_5S_3 - S_{12}S_4S_5 + C_{12}C_3C_4S_5 \\ S_{12}C_5S_3 + C_{12}S_4S_5 + S_{12}C_3C_4S_5 \\ -C_3C_5 + C_4S_3S_5 \\ 0 \end{bmatrix} \tag{10}$$

$$T_6[*][3] = \begin{bmatrix} a_2C_{12} + a_1C_1 - d_4C_{12}S_3 + d_3S_{12} \\ a_2S_{12} + a_1S_1 - d_4S_{12}S_3 - d_3C_{12} \\ d_1 + d_4C_3 \\ 1 \end{bmatrix}$$

Equations (8)-(11) specify the four column **vectors** of the T_6 matrix and hence the Forward solution **of the** manipulator is completely determined. **The** Reverse solution of the manipulator is derived in the next section.

4. The Reverse solution

We usually know the **moves** of the end-effector in **terms** of the T_6 matrices and **it** is required to obtain the values of the joint variables corresponding to **a** given T_6 **matrix. The** closed-form analytical expressions for the joint variables, in terms of the elements of the T_6 matrix, is obtained by isolating each variable by pre-multiplication by a number of **the transforms** *in* **7.**

Let the given T_6 **matrix** be specified **as:**

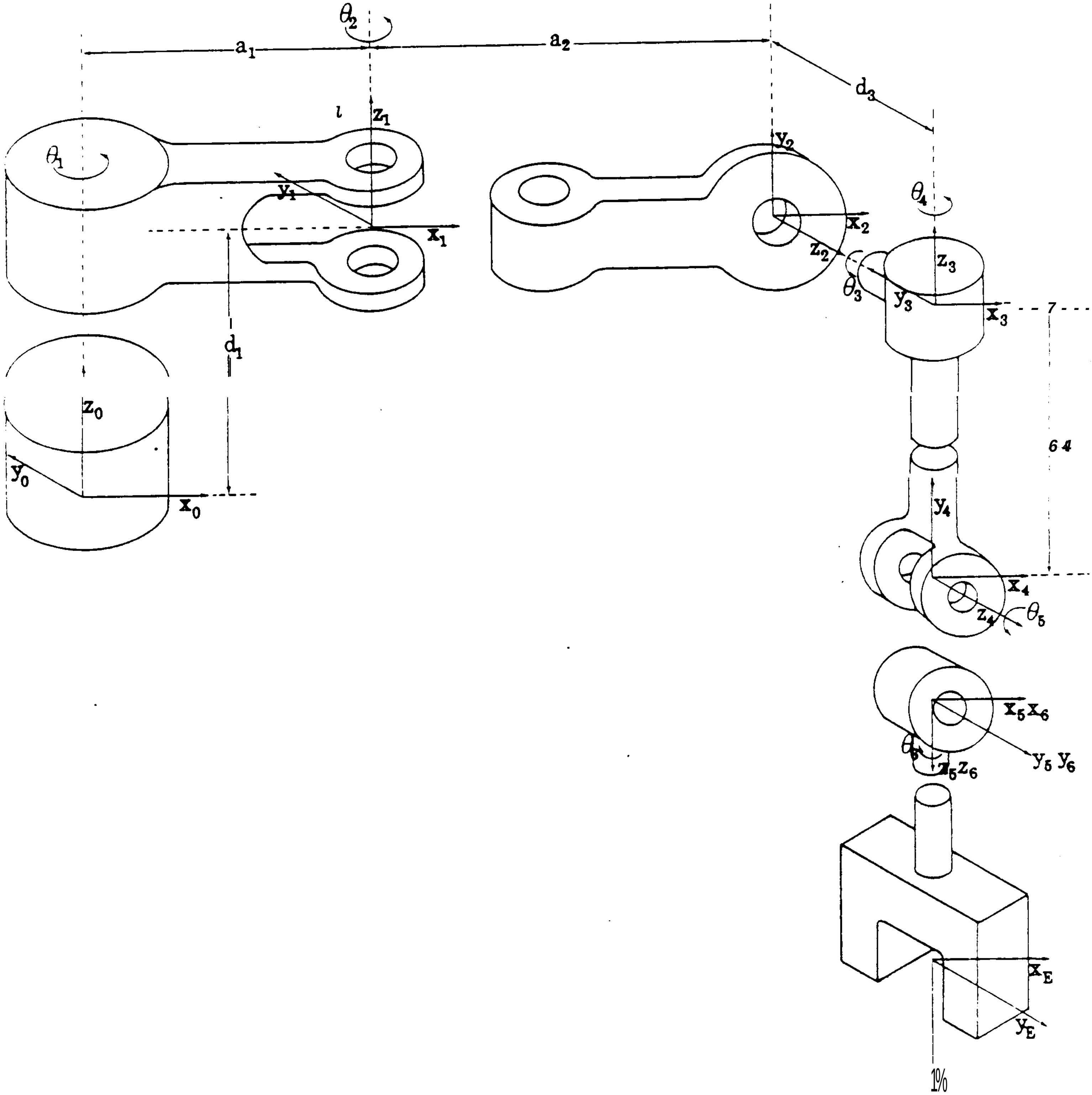

Figure 3-1: Link Coordinates of DDArm II (at the home position).

$$T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Icft-hand-side of (12) is completely specified by (8)-(11). (12)

(a) solution for θ_3:

Comparing the (3.4) clements on **both sidcs of (12) we obtain** *the* **following cquation:**

$$p_z = d_1 + d_4 C_3.$$

. Solving for **C,** **and S_3 the following cxpressions are obtained:**

$$C_3 = \frac{p_z - d_1}{d_4}$$

and

$$S_3 = \pm(1 - C_3^2)^{1/2}.$$

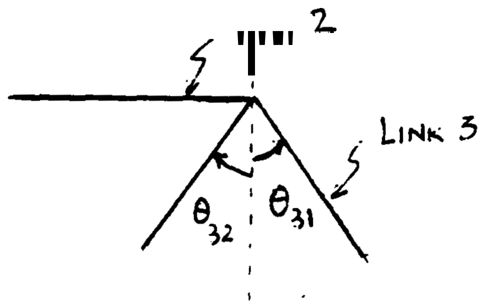

Figure 4-1: Physical Interpretation of Multiple Solutions for 8,

Therefore the two values of 8, are:

$$\theta_{31} = \text{atan2}(S_3, C_3)$$

or

$$\theta_{32} = \text{atan2}(-S_3, C_3) = -\theta_{31}$$

The two **values** of **8, correspond to the** ***elbow out*** **and** ***elbow in*** **positions of the manipulator are arc depicted in** Figure 2. The **correct value of 8, is selected from the above two values based on some criteria. In the**

present program the correct value is selected by the user by specifying *elbow out* or *elbow in*.

(b) solution for θ_2:

Comparing the (1,4) elements of (12) we get

$$-d_4C_{12}S_3 + a_2C_{12} + a_1C_1 + d_3S_{12} = p_x \tag{13}$$

and comparing (2,4) elements of (12) we get

$$-d_4S_{12}S_3 + a_2S_{12} + a_1S_1 - d_3C_{12} = p_y \tag{14}$$

Now let $-d_4S_3 = d'_4$ and $d'_4 + a_2 = \mathbf{d}$. Therefore, (13) and (14) reduce to

$$d'C_{12} + a_1C_1 + d_3S_{12} = p_x \tag{15}$$

$$d'S_{12} + a_1S_1 - d_3C_{12} = p_y \tag{16}$$

Squaring (15) and (16) and adding, we obtain

$$d'C_2 + d_3S_2 = \frac{p_x^2 + p_y^2 - d'^2 - a_1^2 - d_3^2}{2a_1} = A_0$$

Upon substituting $d = rS_\varphi$ and $d_3 = rC_\varphi$, in the above equation, we obtain the expression for 8,:

$$\theta_2 = \mathrm{atan2}[\frac{A_0}{\pm(r^2 - A_0^2)^{0.5}}] - \mathrm{atan2}[\frac{d'}{d_3}]$$

The two values of 8, correspond to the *right* and *left* shoulder configurations of the manipulator and are depicted in Figure 3. The correct value of 8, must be selected based on some criteria. In the present program the user selects **this** value by specifying the right-shoulder or *left-shoulder* configuration.

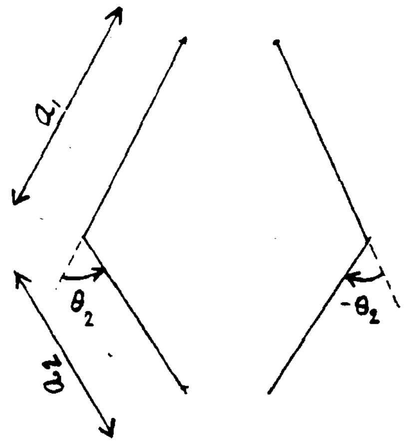

Figure 4-2: Physical Interpretation of Multiple Solutions for 8,.

(c) solution for θ_1:

Multiplying (4) by S_{12} and (5) by $-C_{12}$, we obtain:

$$p_x S_{12} - P_y C_{12} = a_1 S_2 + d_3$$

Upon substituting $p_x = lC_\varphi$ and $P_y = lS_\varphi$, in the above equation. the expression for θ_2 is obtained as:

$$\theta_1 = \mathrm{atan2}[\frac{a_1 S_2 + d_3}{(l^2 - (a_1 S_2 + d_3)^2)^{0.5}}] - \theta_2 + \mathrm{atan2}[\frac{p_y}{p_x}]$$

The two values for 8, correspond to the left and right shoulder configurations. Having chosen the correct value of θ_2, the value of θ_1 is unique.

(d) solution for 8,: Having obtained the values of θ_1.θ_2 and θ_3 it now remains to find the values of θ_4,θ_5 and 8,. Premultiplying both sides of (1) by $({}^0T_3)^{-1}$ we get

$$({}^0T_3)^{-1}T_6 = {}^4T_6.$$

In expanded form the above equation is written as:

$$\begin{bmatrix} C_{12}C_3 & C_3S_{12} & S_3 & - \\ -S_{12} & C_{12} & 0 & - \\ -C_{12}S_3 & -S_{12}S_3 & C_3 & - \\ (& 0 & 0 & 1 \end{bmatrix} T_6 = \begin{bmatrix} S_4S_6 + C_4C_5C_6 & C_6S_4 - C_4C_5S_6 & C_4S_5 & 0 \\ -C_4S_6 + C_5C_6S_5 & -C_4C_6 - C_5S_4S_6 & S_4S_5 & 0 \\ S_5C_6 & -S_5S_6 & d_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Comparing the (3,3) elements on both sides of (17) we obtain: (17)

$$-C_{12}S_3a_x - S_{12}S_3a_y + C_3a_z = -C_5$$

or

$$C_5 = C_{12}S_3a_x + S_{12}S_3a_y - C_3a_z.$$

Comparing the (1.3) and (2,3) elements on both sides of (17)

$$C_4S_5 = C_{12}C_3a_x + C_3S_{12}a_y + S_3a_z = A$$

$$S_4S_5 = -S_{12}a_x + C_{12}a_y = B \qquad (18)$$

Squaring and adding (18) and (19) we obtain the expression for S_5 as (19)

$$S_5 = \pm(A^2 + B^2)^{0.5}$$

Evaluating θ_5 using the double argument atan2 function the following two expressions are obtained:

$$\theta_{51} = \mathrm{atan2}(S_5, C_5)$$

or

$$\theta_{52} = \mathrm{atan2}(-S_5, C_5).$$

As in the case of θ_1 and θ_3, the correct value of θ_5 is chosen based on some criterion. In the present case this is selected by the user.

(e) solution for θ_4:

Upon comparing the (1.3) and (2.3) elements on both sides of (17) we get:

$$C_4S_5 = C_{12}C_3a_x + C_3S_{12}a_y + S_3a_z$$

$$S_4S_5 = -S_{12}a_x + C_{12}a_y.$$

Therefore,

$$\theta_4 = \mathrm{atan2}(S_4S_5, C_4S_5) \qquad \text{if} \quad \theta_5 > 0$$

or

$$\theta_4 = \theta_4 + \pi \qquad \text{if} \quad \theta_5 < 0$$

The manipulator becomes degenerate when $\theta_5 = 0$.

(f) solution for θ_6:

Upon comparing the (3.1) and (3.2) elements on both sides of (17), the following equations are obtained

$$C_6S_5 = -C_{12}S_3n_x - S_{12}S_3n_y + C_3n_z$$

$$-S_6S_5 = -C_{12}S_3o_x - S_{12}S_3o_y + C_3o_z.$$

$$\theta_6 = \operatorname{atan2}(-S_5S_6, S_5C_6) \qquad \text{if} \qquad \theta_5 > 0$$

$$\theta_6 = \theta_6 + \pi \qquad \text{if} \qquad \theta_5 < 0$$

When 8, = 0 the manipulator is degenerate and only the sum of $(\theta_4 + \theta_6)$ is important. At this point one of the angles is given an arbitrary value (usually the present value) and the other computed accordingly.

(g) Solution for 8, and 8, when 8, = 0: .

When 8, = 0, $C_5 = 1$. Comparing the (1.1) of (1.2) elements on both sides of (6) **wc** get

$$\sin(\theta_4 - \theta_6) = C_{12}C_3o_x + C_3S_{12}o_y + S_3o_z = A$$

and

$$\cos(\theta_4 - \theta_6) = C_{12}C_3n_x + C_3S_{12}n_y + S_3n_z = B$$

Thus the value of $(\theta_4 - \theta_6)$ is

$$(\theta_4 - \theta_6) = \operatorname{atan2}(A, B)$$

At this point 8, or 8, is given an arbitrary value (usually the present value) and the other computed accordingly.

The analytical expressions For the six joints of the DDArm **II are** outlined in paragraphs (a)-(g). The multiple solutions For **joints** 2, 3 and 5 **give rise to** 8 sets of Reverse solutions. These are represented diagrammatically in **Figure 4.**

5. The Jacobian

In a manipulator, differential changes in position and orientation of T, are caused by differential **changes** (dq_i) in the joint coordinates. The **transformation** relating the differential changes in **the** joint coordinates to the differential changes in the T_6 frame is called the Jacobian **(a** 6 x 6 **matrix). Each** column of the Jacobian corresponds to a differential translation and rotation **vector** corresponding to the differential change in each joint coordinate.

The elements of the 6 column vectors of the Jacobian mamx are:

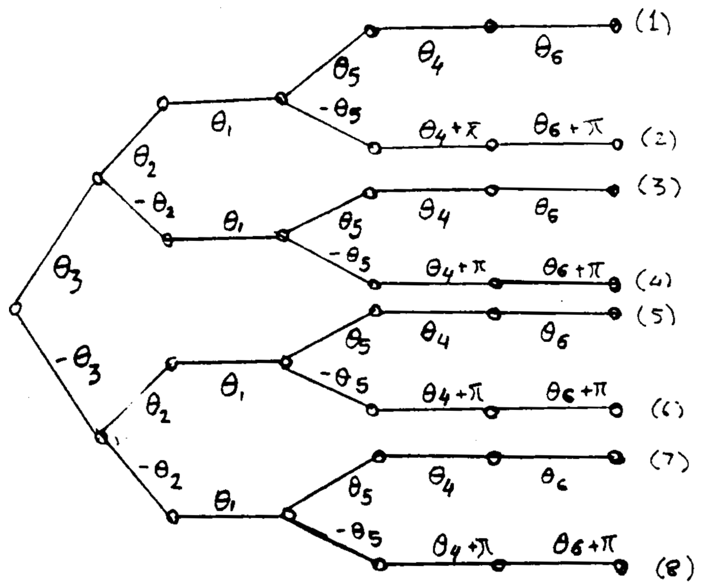

Figure 4-3: Graph Depicting Multiple Sets of Reverse Solutions

$$d_{1x} = -(-C_5C_6S_{12}S_4 + C_4S_{12}S_6 + C_{12}C_3C_4C_5C_6 + C_{12}C_3S_4S_6 - C_{12}C_6S_3S_5)py1 + (C_3C_4C_5C_6S_{12} + C_3S_{12}S_4S_6 - C_6S_{12}S_3S_5 + C_{12}C_5C_6S_4 - C_{12}C_4S_6)px1;$$

$$d_{1y} = -(C_5S_6S_{12}S_4 + C_4S_{12}C_6 - C_{12}C_3C_4C_5S_6 + C_{12}C_3S_4C_6 + C_{12}S_6S_3S_5)py1 + (-C_3C_4C_5S_6S_{12} + C_3S_{12}S_4C_6 + S_6S_{12}S_3S_5 - C_{12}C_5S_6S_4 - C_{12}C_4C_6)px1;$$

$$d_{1z} = -(-S_{12}S_4S_5 + C_{12}C_3C_4S_5 + C_{12}C_5S_3)\,py1 + (C_3C_4S_{12}S_5 + C_5S_{12}S_3 + C_{12}S_4S_5)px1;$$

$$\delta_{1x} = C_3C_6S_5 + S_3S_4S_6 + C_4C_5C_6S_3;$$

$$\delta_{1y} = -C_3S_5S_6 + C_6S_3S_4 - C_4C_5S_3S_6;$$

$$\delta_{1z} = -C_3C_5 + C_4S_3S_5;$$

$$d_{2x} = -(C_4S_2S_6 - C_5C_6S_2S_4 - C_2C_6S_3S_5 + C_2C_3S_4S_6 + C_2C_3C_4C_5C_6)py2 + (-C_2C_4S_6 + C_2C_5C_6S_4 - C_6S_2S_3S_5 + C_3S_2S_4S_6 + C_3C_4C_5C_6S_2)px2;$$

$$d_{2y} = -(C_4S_2C_6 + C_5S_6S_2S_4 + C_2C_6S_3S_5 + C_2C_3S_4C_6$$
$$- C_2C_3C_4C_5S_6)py2 + (-C_2C_4C_6 - C_2C_5S_6S_4 + S_6S_2S_3S_5$$
$$+ C_3S_2S_4C_6 - C_3C_4C_5S_6S_2)px2;$$

$$d_{2z} = -(-S_2S_4S_5 + C_2C_5S_3 + C_2C_3C_4S_5)py2 + (C_2S_4S_5 + C_5S_2S_3 +$$
$$C_3C_4S_2S_5)px2;$$

6, $= C_3C_6S_5 + S_3S_4S_6 + C_4C_5C_6S_3;$

$$\delta_{2y} = -C_3S_5S_6 + C_6S_3S_4 - C_4C_5S_3S_6;$$

$$\delta_{2z} = -C_3C_5 + C_4S_3S_5;$$

$$d_{3x} = -(C_6S_3S_5 + C_3S_4S_6 + C_3C_4C_5C_6)py3 + (C_3C_6S_5 + S_3S_4S_6 +$$
$$C_4C_5C_6S_3)px3;$$

$$d_{3y} = -(S_6S_3S_5 + C_3S_4C_6 - C_3C_4C_5S_6)py3 + (-C_3S_6S_5 + S_3S_4C_6 -$$
$$C_4C_5S_6S_3)px3;$$

$$d_{3z} = -(C_5S_3 + C_3C_4S_5)py3 + (-C_3C_5 + C_4S_3S_5)px3;$$

$$\delta_{3x} = C_4S_6 - C_5C_6S_4;$$

$$\delta_{3y} = C_4C_6 + C_5S_4S_6;$$

$$\delta_{3z} = -S_4S_5;$$

$$d_{4x} = 0;$$

$$d_{4y} = 0;$$

$$d_{4z} = 0;$$

$$\delta_{4x} = C_6S_5;$$

$$\delta_{4y} = -S_5S_6;$$

$$\delta_{4z} = -C_5;$$

$d_{5x} = 0;$

$d_{5y} = 0;$

$d_{5z} = 0;$

$\delta_{5x} = S_6;$

$\delta_{5y} = C_6;$

$\delta_{5z} = 0;$

$d_{6x} = 0;$

$d_{6y} = 0;$

$d_{6z} = 0;$

$\delta_{6x} = 0;$

$\delta_{6y} = 0;$

$\delta_{6z} = 1;$

where,

$px_1 = -d_4C_{12}S_3 + a_2C_{12} + a_1C_1 + d_3S_{12}$

$py_1 = -d_4S_{12}S_3 + a_2S_{12} + a_1S_1 - d_3C_{12}$

$px_2 = a_2C_2 - d_4C_2S_3 + d_3S_2$

$py_2 = a_2S_2 - d_4S_2S_3 - d_3C_2$

$px_3 = -d_4S_3$

$py_3 = d_4C_3.$

A MATRIX PARAMETER

link	variable	θ	*a*	a	d
1	θ_1	θ_1	0°		d_1
2	θ_2	θ_2	90°	a_2	0
3	θ_3	θ_3	-90°	0	d_3
4	θ_4	θ_4	90°	0	d_4
5	θ_5	θ_5	90°	0	0
6	θ_6	θ_6	0°	0	0

Table 1

References

[1] Denavit, J. and Hartenberg, R. S.
A Kinematic Notation for Lower Pair Mechanisms Based on Matrices.
Journal of *Applied Mechanics* 77(2):215-221, June, 1955.

[2] J. J. Murray and C. P. Neuman.
ARM: An Algebraic Robot Dynamic Modeling Program.
1983.
Unpublished.

[3] Paul, R.P.
Robot Manipulators: Mathematics, Programming and Control.
MIT Press, Cambridge, MA, 1981.