Proof: Theorem 4.1 follows directly from Theorem 3.1 and the details of the proof are omitted.

Corollary 4.2: Given the system (19) with two-control agents, assume that

(1) $(A_{11}(s),\ B_1(s))$ and $(A_{22}(s),\ B_2(s))$ are both controllable;

(2) $\left\{\begin{bmatrix} A_{11}(s) & A_{12}(s) \\ A_{21}(s) & A_{22}(s) \end{bmatrix},\ \begin{bmatrix} B_1(s) & 0 \\ 0 & B_2(s) \end{bmatrix}\right\}$ is controllable.

Then the system has no decentralized fixed modes.

Note that the conjecture in [7] is still correct for the interconnected system described in differential operator form.

Example:

$$\begin{bmatrix} s^2-1 & 2 \\ -3 & s-4 \end{bmatrix}\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} s+11 \\ 0 \end{bmatrix} u_1(s) + \begin{bmatrix} 0 \\ s-5 \end{bmatrix} u_2(s). \quad (20)$$

Applying Corollary 4.2 to (20), we know

$(A_{11}(s),\ B_1(s)) = (s^2-1,\ s+11)$

and $(A_{22}(s),\ B_2(s)) = (s-4,\ s-5)$ are both controllable;

$$(A(s),\ B(s)) = \left(\begin{bmatrix} s^2-1 & 2 \\ -3 & s-4 \end{bmatrix},\ \begin{bmatrix} s+11 & 0 \\ 0 & s-5 \end{bmatrix}\right) \text{ is controllable.}$$

According to Corollary 4.2, system (20) has no fixed modes.

It is easily seen that Theorem 8 in [4] cannot be applied for this example. Theorem 3.1 in [2] can be used, but it requires more computations. In this context, Theorem 4.1 and Corollary 4.2 may be more useful.

V. Conclusions

A study of decentralized fixed modes of the system described in differential operator form has been presented in this paper. The following results have been obtained. 1) Frequency domain characterization for the existence of fixed modes of a linear system described in differential operator form (Theorem 3.1); 2) A new characterization for the existence of fixed modes of a system consisting of N scalar subsystems, and described in differential operator form (Theorem 3.2); 3) A necessary and sufficient condition for an interconnected system described in differential operator form to have no fixed modes (Theorem 4.1).

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Acceleration-Constrained Time-Optimal Control in n Dimensions

DAI FENG AND BRUCE H. KROGH

Abstract—**The minimum-time acceleration of a generalized point mass from an initial position and velocity to the origin in n dimensions is solved by transforming the problem to an equivalent problem in two dimensions and analytically integrating the system differential equations. Computation of the optimal control is thereby reduced to the solution of five simultaneous nonlinear equations. A *numerical continuation method* is presented for solving these equations by starting at the known solution of a related single-dimensional problem and progressing incrementally to the desired solution. The problem and solution method are illustrated by a numerical example.**

I. Introduction

This note concerns the time-optimal acceleration control of a generalized point mass from an initial position and velocity to the origin of the state space. The double-integrator system dynamics are motivated by applications where kinematic trajectories are computed as reference signals for lower-level servocontrollers. Applications include the supervisory control of robotic manipulators [1] and trajectory optimization for missile guidance [2] where double-integrators provide a simple model of the system dynamics for trajectory planning. In certain applications so-called *external linearization* via state transformation and nonlinear feedback leads to an *exact* representation of the nonlinear systems dynamics (of order $2n$) by an equivalent linear system consisting of n uncoupled double-integrators [3], [4].

We consider the problem of driving the state of n double integrators to the origin when the magnitude of the control (acceleration) vector is constrained. It is this control constraint which couples the double integrators. Indeed, if the limits on each component of the acceleration vector were independent, the problem would reduce to n independent time-optimal control problems each solved by the well-known bang-bang control [5]. However, in many applications the control limits for each double integrator are not independent. For example, if the double-integrator model represents the dynamics of a robot end-effector in Cartesian coordinates for the purpose of computing minimum-time trajectories, Cartesian acceleration constraint set is a complex state-dependent function of the physical joint torque limits. As a model simplification, constraining the magnitude rather than components of the Cartesian acceleration vector leads to smoother control action. Methods for computing appropriate acceleration constraints for particular applications are currently under investigation [6].

In this correspondence we present the solution of this time-optimal control problem as interesting in its own right for two reasons. First, it is a contribution to the small class of optimal control problems for which numerical solution of the two-point boundary value problem can be avoided by analytically integrating the system equations. Second, a *numerical continuation method* is proposed for solving the resulting nonlinear equations by starting with the known solution to a related single-dimension double-integrator problem and progressing incrementally to the desired solution. Properties of the time-optimal control are illustrated by an example.

II. Problem Formulation

Consider the system

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} O & I \\ O & O \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} O \\ I \end{pmatrix} r \quad (1)$$

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where $\boldsymbol{w}(t)$, $\boldsymbol{z}(t)$, $\boldsymbol{r}(t)$ are the position, velocity, and acceleration, respectively, of a generalized point mass in n-dimensional space and O, I are $n \times n$ zero and identity matrices, respectively. The magnitude of the acceleration control $\boldsymbol{r}(t)$ is constrained as

$$\|r(t)\| \leq \bar{r}. \tag{2}$$

Without loss of generality, we assume $\bar{r}$ to be unity. Admissible controls $\boldsymbol{r}$ are piecewise right-continuous vector-functions of time satisfying (2).

The control objective is to transfer the system from a given initial state $(\boldsymbol{w}_o, \boldsymbol{z}_o)$ to rest at the origin $(\boldsymbol{w}_f = \boldsymbol{0}, \boldsymbol{z}_f = \boldsymbol{0})$ in minimum time. That is, we wish to find $\boldsymbol{r}^*$ to minimize

$$T = \int_0^T dt$$

subject to (1) and (2) with boundary conditions

$$(w(0),\ z(0)) = (w_o,\ z_o),\ (w(T),\ z(T)) = (\mathbf{0},\ \mathbf{0}). \tag{3}$$

Letting $(\boldsymbol{g}(t), \boldsymbol{h}(t))$ be the costate with $\boldsymbol{g}(t), \boldsymbol{h}(t) \in R^n$, the Hamiltonian H is

$$H = 1 + g'(t)z(t) + h'(t)r(t) \tag{4}$$

where the arguments of H are implied from the problem formulation and $'$ denotes transpose. From the minimum principle the optimal costate trajectory satisfies

$$\dot{g} = -\frac{\partial H}{\partial w} = \mathbf{0} \tag{5}$$

and

$$\dot{h} = -\frac{\partial H}{\partial z} = -g, \tag{6}$$

and the optimal control $\boldsymbol{r}^*(t)$ is given by

$$r^*(t) = -\frac{h(t)}{\|h(t)\|} \quad \text{for } h(t)) \neq \mathbf{0}. \tag{7}$$

Due to the compactness of the control constraint set for this time-optimal control problem, the above necessary conditions are also sufficient and the optimal control exists and is unique [7]. From (5) and (6), $\boldsymbol{g}(t)$ is constant, $\boldsymbol{h}(t)$ is affine in time, and it can be easily shown that there are no singular intervals [5]. From (7), $\boldsymbol{r}^*(t)$ is a unit vector in the direction $-\boldsymbol{h}(t)$ (for $\boldsymbol{h}(t) \neq \boldsymbol{0}$). Since the trajectory of $\boldsymbol{h}(t)$ is contained in a line in R^n, $\boldsymbol{r}^*(t)$ remains in the plane containing the origin and $\boldsymbol{h}(t)$ for $t \in (0, T)$. For the degenerate case where the line containing the trajectory of $\boldsymbol{h}(t)$ passes through the origin, $\boldsymbol{r}^*(t)$ is the single-switch bang-bang control for the equivalent simple double-integrator problem. These observations lead to the following proposition which permits us to reduce the problem in n dimensions to a problem in the space spanned by the initial position and velocity vectors.

Proposition: Let $\boldsymbol{r}^*(t)$ be the time-optimal control for (1) with boundary conditions (3) and let $P = \text{span}\,\{\boldsymbol{r}^*(t) | 0 \leq t \leq T\}$ and $M = \text{span}\,\{\boldsymbol{w}_o, \boldsymbol{z}_o\}$, then $P = M$ and $\boldsymbol{g}(t), \boldsymbol{h}(t) \in P$ for $0 \leq t \leq T$.

Proof: We first show that $\boldsymbol{z}_o \in P$. Suppose $\boldsymbol{z}_o \notin P$. This implies $\text{proj}\,\{\boldsymbol{z}_o, P^p\} \neq 0$, where $\text{proj}\,\{\boldsymbol{a}, S\}$ denotes the projection of vector $\boldsymbol{a}$ onto the subspace S, and S^p denotes the subspace orthogonal to a given subspace S. But $\text{proj}\,\{\dot{\boldsymbol{z}}(t), P^p\} = \text{proj}\,\{\boldsymbol{r}^*(t), P^p\} = \boldsymbol{0}$ for all $0 \leq t \leq T$, which implies $\text{proj}\,\{\boldsymbol{z}(t), P^p\} = \text{proj}\,\{\boldsymbol{z}_o, P^p\} \neq \boldsymbol{0}$ for all $0 \leq t \leq T$. Thus, $\boldsymbol{z}(T) \neq \boldsymbol{0}$ contradicting the boundary condition. Similarly, since $\boldsymbol{z}(t) \in P$ for all $0 \leq t \leq T$, $\dot{\boldsymbol{w}}(t) = \boldsymbol{z}(t)$, and $\boldsymbol{w}(T) = \boldsymbol{0}$, it is necessary that $\boldsymbol{w}_o \in P$. Hence, $M \subset P$.

Since P is at most of dimension two, linear independence of $\boldsymbol{w}_o$ and $\boldsymbol{z}_o$ implies that $P = \text{span}\,\{\boldsymbol{w}_o, \boldsymbol{z}_o\} = M$. If $\boldsymbol{w}_o$ and $\boldsymbol{z}_o$ are colinear, $\boldsymbol{r}^*(t)$ is the well-known bang-bang control [5] which is colinear with $\boldsymbol{w}_o$ and $\boldsymbol{z}_o$. Thus, $\dim\,\{P\} = 1$, implying $P = \text{span}\,\{\boldsymbol{w}_o, \boldsymbol{z}_o\} = M$.

It follows from the definition of P and (5)–(7) that $\boldsymbol{g}(t), \boldsymbol{h}(t) \in P$ for $0 \leq t \leq T$. □

This proposition implies that the optimal control in R^n is the solution to an equivalent minimum-time problem in the space M spanned by $\boldsymbol{w}_o$ and $\boldsymbol{z}_o$. For the nontrivial case when M is of dimension two, the mapping into an equivalent two-dimensional problem is not unique. When $n = 3$ ($\boldsymbol{w}(t)$ is the system position in three-dimensional Euclidean space, for example) initial conditions for an equivalent problem in R^2 are given by mapping $\boldsymbol{w}_o$ and $\boldsymbol{z}_o$ from R^3 into R^2 with the transformation

$$\tau = \begin{bmatrix} -\sin\xi & \cos\xi & 0 \\ \sin\psi\cos\xi & \sin\psi\sin\xi & \cos\psi \end{bmatrix} \tag{8}$$

where

$$\xi = \tan^{-1}(y_2/y_1),\ \psi = \tan^{-1}(y_3/\sqrt{y_1^2 + y_2^2})$$

with $\boldsymbol{y} = [y_1, y_2, y_3]' = \boldsymbol{w}_o \times \boldsymbol{z}_o$. The optimal control in two-dimensions is mapped back into R^3 by the transformation $\Lambda = \tau'$.

Having reduced the general problem in n dimensions to an equivalent problem in the space M, Sections III and IV concern the solution of the minimum-time problem in R^2.

III. Analytical Solution for Two Dimensions

Let $\boldsymbol{x}(t)$, $\boldsymbol{v}(t)$, $\boldsymbol{u}(t)$ be the position, velocity, and acceleration, respectively, of a point mass with dynamics given by (1) for the particular case $n = 2$. Since $\boldsymbol{u}^*(t)$ is a unit vector (7), it is uniquely determined in R^2 by its angle $\beta^*(t)$ in polar coordinates. Thus, the time-optimal control problem can be reformulated in terms of the single control variable $\beta(t)$ with $\boldsymbol{u}(t) = [\cos\beta(t), \sin\beta(t)]'$. In terms of $\beta(t)$ the Hamiltonian is given by

$$H = 1 + p'(t)v(t) + [q_1(t)\cos\beta(t) + q_2(t)\sin\beta(t)]$$

and the necessary conditions for the optimal costate trajectory remain as in (5), (6), where $\boldsymbol{p}(t), \boldsymbol{q}(t) \in R^2$ correspond to the costate vectors $\boldsymbol{g}(t)$, $\boldsymbol{h}(t)$, respectively, for $n = 2$. Since $\beta(t)$ is unbounded, the minimum principle implies

$$\frac{\partial H}{\partial \beta(t)} = -q_1(t)\sin\beta(t) + q_2(t)\cos\beta(t) = 0. \tag{9}$$

The Hamiltonian does not depend explicitly on t, giving the additional necessary condition

$$1 + p'(t)v(t) + [q_1(t)\cos\beta(t) + q_2(t)\sin\beta(t)] \equiv 0. \tag{10}$$

Let c_1, c_2, c_3, and c_4 be the initial values for p_1, p_2, q_1, and q_2, respectively. Equations (5), (6), (9) imply the so-called bilinear tangent law [2] for the optimal angle

$$\tan\beta^*(t) = \frac{c_2 t - c_4}{c_1 t - c_3} \tag{11}$$

and the components $u_1^*(t)$, $u_2^*(t)$ of the optimal control $\boldsymbol{u}^*(t)$ are given by

$$u_1^*(t) = \frac{c_1 t - c_3}{\sqrt{[c_1^2 + c_2^2]t^2 - 2[c_1c_3 + c_2c_4]t + c_3^2 + c_4^2}}$$

$$u_2^*(t) = \frac{c_2 t - c_4}{\sqrt{[c_1^2 + c_2^2]t^2 - 2[c_1c_3 + c_2c_4]t + c_3^2 + c_4^2}}. \tag{12}$$

Using these general expressions for the components of the optimal control (12), the differential equations (1) were integrated analytically (with the aid of integration tables [8]) to obtain expressions for the optimal state trajectory $(\boldsymbol{x}^*(t), \boldsymbol{v}(t))$ in terms of the unknown constants c_1, c_2, c_3, c_4. Applying the boundary conditions (3) and the necessary condition (10) (at $t = T$) yields the following simultaneous nonlinear equations involving c_1, c_2, c_3, c_4, and T[1]:

$$\frac{1}{c^{3/2}}\left[\frac{1}{2}\left(c_3+\frac{c_1b}{2c}\right)(2cT+b)-\frac{c_1}{2k}\right]\sinh^{-1}\left(\frac{2cT+b}{\sqrt{q}}\right)$$
$$-\frac{1}{c^2}\left[\frac{c_1}{4}(2cT+b)\sqrt{cT^2+bT+a}\right.$$
$$\left.+\sqrt{c}\left(c_3+\frac{c_1b}{2c}\right)\sqrt{c^2T^2+bcT+ac}\right]$$
$$-\left[\frac{1}{c^{1/2}}\left(c_3+\frac{c_1b}{2c}\right)\sinh^{-1}\left(\frac{b}{\sqrt{q}}\right)-\frac{c_1}{c}\sqrt{a}\right]T$$
$$-\frac{1}{c^{3/2}}\left[\frac{b}{2}\left(c_3+\frac{c_1b}{2c}\right)-\frac{c_1}{2k}\right]\sinh^{-1}\left(\frac{b}{\sqrt{q}}\right)$$
$$+\frac{1}{c^2}\left[\frac{c_1}{4}b\sqrt{a}+c\sqrt{a}\left(c_3+\frac{c_1b}{2c}\right)\right]+x_3^0T+x_1^0=0 \tag{13}$$

$$\frac{1}{c^{3/2}}\left[\frac{1}{2}\left(c_4+\frac{c_2b}{2c}\right)(2cT+b)-\frac{c_2}{2k}\right)\sinh^{-1}\left(\frac{2cT+B}{\sqrt{q}}\right)$$
$$-\frac{1}{c^2}\left[\frac{c_2}{4}(2cT+b)\sqrt{cT^2+bT+a}\right.$$
$$\left.+\sqrt{c}\left(c_4+\frac{c_2b}{2c}\right)\sqrt{c^2T^2+bcT+ac}\right]$$
$$-\left[\frac{1}{c^{1/2}}\left(c_4+\frac{c_2b}{2c}\right)\sinh^{-1}\frac{b}{\sqrt{q}}-\frac{c_2}{c}\sqrt{a}\right]T$$
$$-\frac{1}{c^{3/2}}\left[\frac{b}{2}\left(c_4+\frac{c_2b}{2c}\right)-\frac{c_2}{2k}\right]\sinh^{-1}\left(\frac{b}{\sqrt{q}}\right)$$
$$+\frac{1}{c^2}\left[\frac{c_2}{4}b\sqrt{a}+\sqrt{ac}\left(c_4+\frac{c_2b}{2c}\right)\right]$$
$$+x_4^oT+x_2^o=0 \tag{14}$$

$$\frac{1}{c^{1/2}}\left[c_3+\frac{c_1b}{2c}\right]\sinh^{-1}\left(\frac{2cT+b}{\sqrt{q}}\right)-\frac{c_1}{c}\sqrt{cT^2+bT+a}$$
$$-\frac{1}{c^{1/2}}\left(c_3+\frac{c_1b}{2c}\right)\sinh^{-1}\left(\frac{b}{\sqrt{q}}\right)+\frac{c_1}{c}\sqrt{a}+x_3^o=0 \tag{15}$$

$$\frac{1}{c^{1/2}}\left[c_4+\frac{c_2b}{2c}\right]\sinh^{-1}\left(\frac{2cT+b}{\sqrt{q}}\right)-\frac{c_2}{c}\sqrt{cT^2+bT+a}$$
$$-\frac{1}{c^{1/2}}\left(c_4+\frac{c_2b}{2c}\right)\sinh^{-1}\left(\frac{b}{\sqrt{q}}\right)+\frac{c_2}{c}\sqrt{a}+x_4^o=0 \tag{16}$$

$$1-(c_3-c_1T)^2-(c_4-c_2T)^2=0 \tag{17}$$

where $a = c_3^2 + c_4^2$; $b = -2(c_1c_3 + c_2c_4)$; $c = c_1^2 + c_2^2$; $q = 4ac - b^2$, and $k = 4c/q$. Thus, computation of the time-optimal control $u^*(t)$ (12) is reduced to the problem of solving (13)–(17) numerically.

IV. Numerical Solution

Our initial attempts to solve (13)–(17) numerically using standard computer routines [9] practically never converged (except, of course, for cases where an initial guess sufficiently close to the solution could be computed by other means). On the other hand, for the trivial case when $x_o = \kappa v_o$ with $\kappa \geq 0$ (M one-dimensional) the solution to (13)–(17) is given analytically by

$$c_1=1,\quad c_2=0,\quad c_3=c_1t_s,\quad c_4=0,\quad T=\|v_o\|+\sqrt{2\|v_o\|^2+4\|x_o\|} \tag{18}$$

where t_s, the switch time for the optimal bang-bang control, is given by

$$t_s=\sqrt{\|x_o\|+0.5\|v_o\|^2}-\|v_o\|. \tag{19}$$

To circumvent the numerical difficulties in solving (13)–(17) for arbitrary x_o, v_o, we implemented a *numerical continuation method* which begins from the known solution (18) for a related one-dimensional problem and proceeds to solve (13)–(17) for a sequence of initial conditions which incrementally approach x_o, v_o. Since the solution to (13)–(17) for each set of initial conditions is near the solution from the previous iteration, convergence problems are avoided.

To define the sequence of equations to be solved, the initial conditions are parameterized by α for $0 \leq \alpha \leq 1$ as

$$x_o(\alpha)=\|x_o\|[\cos(\alpha\phi),\ \sin(\alpha\phi)]',$$
$$v_o(\alpha)=\|v_o\|[\cos(\alpha\theta),\ \sin(\alpha\theta)]' \tag{20}$$

where ϕ, θ are the angles of the vectors x_o, v_o, respectively, in polar form.

Starting at $\alpha = 0$, the parameterized initial conditions $x_o(0)$, $v_o(0)$ are colinear and the exact solutions for (13)–(17) are given by (18). As α is incremented (13)–(17) are solved by a standard numerical algorithm [9] using the previous solution as the initial guess. This process continues until $\alpha = 1$ for which the parameterized initial conditions $x_o(1)$, $v_o(1)$ are the desired vectors x_o, v_o.

V. Example

Consider the minimum-time problem in two dimensions (and the equivalent class of n-dimensional problems) with initial conditions

$$x_o=[15,\ 10]',\quad v_o=[-1,\ 4]'. \tag{21}$$

Table I shows solutions to (13), (14) for the parameterized initial conditions (20) as α is incremented from 0 to 1. Fig. 1 illustrates the minimum-time trajectory with the direction of the optimal control indicated at time intervals of 0.5 s.

This example illustrates certain qualitative properties of the solution to the acceleration-constrained minimum-time problem. The total variation of the angle of the control vector never exceeds 180°; the bang-bang control being the extreme case. Also, the final acceleration is tangent to the trajectory at the terminal time. From these properties it can be shown that the optimal trajectory never cycles around the terminal point; that is, the total variation in the angle of $x^*(t)$ is less than 180°.

VI. Conclusion

It has been shown that the time-optimal control of the system (1) from an initial state to the origin in n dimensions is equivalent to a two-dimensional problem. This result also holds if the desired terminal velocity is not zero. However, in this case the plane in which the control lies cannot be determined *a priori*. In contrast to the result for zero terminal velocity where the equivalent two-dimensional problem can be found by a transformation such as (8), it can be shown that when $z_f \neq 0$ the desired two-dimensional subspace depends on the final time T which, of course, is not known.

Although the problem considered above is basic and simply stated, to the authors' knowledge its solution has not appeared in the literature. A related problem solved in the book by Bryson and Ho [2] concerns the optimal acceleration of a point mass to a rectilinear path so as to maximize the final speed of the system. For this problem the *bilinear tangent law* (11) reduces to a *linear tangent law* due to the special form of the boundary conditions, and the double integrator dynamics can be easily integrated. This simplification does not occur for our problem.

[1] Equations (13)–(17) have been verified by simulation of numerical examples.

TABLE I
SOLUTIONS FOR PARAMETERIZED INITIAL CONDITIONS

α	x_1	x_2	v_1	v_2	c_1	c_2	c_3	c_4	T
0.0	18.028	0.000	-4.123	0.000	0.194	0.000	0.199	0.000	6.177
0.1	17.997	1.059	-4.087	0.545	0.193	0.036	0.213	0.080	6.215
0.2	17.903	2.115	-3.979	1.081	0.184	0.100	0.239	0.288	6.380
0.3	17.748	3.164	-3.801	1.597	0.140	0.208	0.190	0.777	6.813
0.4	17.531	4.201	-3.557	2.086	0.061	0.299	0.000	1.405	7.622
0.5	17.254	5.224	-3.250	2.537	0.014	0.314	-0.128	1.745	8.631
0.6	16.917	6.229	-2.886	2.945	0.003	0.295	-0.128	1.856	9.625
0.7	16.522	7.213	-2.471	3.300	0.007	0.272	-0.058	1.882	10.528
0.8	16.070	8.171	-2.013	3.598	0.016	0.252	0.042	1.876	11.330
0.9	15.562	9.101	-1.520	3.833	0.027	0.236	0.156	1.855	12.033
1.0	15.000	10.000	-1.000	4.000	0.039	0.221	0.276	1.826	12.641

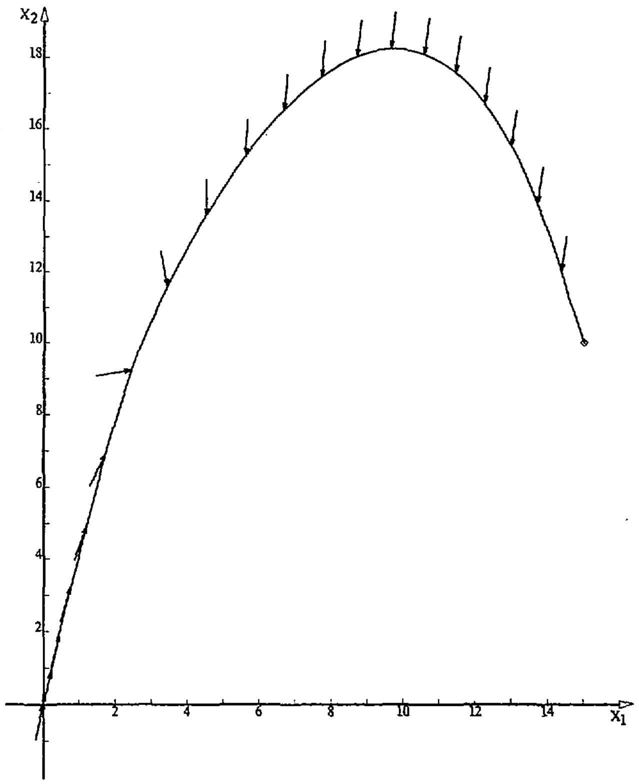

Fig. 1. Time optimal trajectory and control vector.

The reduction of the problem formulated in this note to a set of nonlinear equations makes it feasible to compute the optimal control in real time. Thus, the simple dynamic model (1) could be used as the basis for a feedback control scheme in which the control is recomputed at each sampling instant using the current state. This *open-loop feedback* approach to real-time steering control is considered in [10], [11]. The results of this correspondence are currently being incorporated as a part of the feedback algorithm proposed in [11].

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Discrete Optimal Control with Eigenvalue Assigned Inside a Circular Region

TSU-TIAN LEE AND SHIOW-HARN LEE

Abstract—**A discrete-time optimal control that guarantees that all the closed-loop poles will lie inside a circle centered at (β, 0) with radius α is formulated. It is shown how the exposed problem can be reduced to a standard discrete-time linear quadratic regulator problem. Furthermore, a quantitative measure of the robustness of linear quadratic state feedback design in the presence of a perturbation is obtained. Bounds are derived for allowable nonlinear perturbations such that the resultant closed loop is stable.**

I. INTRODUCTION

For a continuous-time system which is stabilizable and detectable, Anderson and Moore [1] have shown how it is possible to minimize a quadratic performance index and, at the same time, to ensure that the closed-loop system will have poles with real parts all less than some real number α. Similarly, Franklin and Powell [2] have derived a state variable feedback law that minimizes a discrete-time quadratic performance index and, meanwhile, ensures that the closed-loop system has poles all less than $\alpha \leq 1$. The aim of this note is to formulate a discrete-time quadratic minimization problem in such a way as to give rise to a linear state variable feedback law guaranteeing that closed-loop poles all lie inside a circle centered at (β, 0) with radius α, where $\alpha + |\beta| \leq 1$. Moreover, it is known that the stability of a discrete-time linear quadratic regulator is guaranteed. But the behavior of such regulated discrete-time systems to nonlinear perturbations is unknown. We have, therefore, derived bounds for allowable nonlinear perturbations such that the resultant closed-loop is stable.

II. OPTIMIZATION WITH PRESCRIBED CLOSED-LOOP POLES INSIDE A CIRCULAR REGION

Consider a linear time-invariant discrete-time controllable system

$$X(p+1)=AX(p)+BU(p), \qquad X(0)=X_0 \tag{1}$$

where X is an $n \times 1$ state vector, U is an $r \times 1$ control vector, and A and B are $n \times n$ and $n \times r$ constant matrices, respectively.

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