Global Optimum Path Planning for a Redundant Space Robot

Om P. Agrawal* and Yangsheng Xu

CMU-RI-TR-91-15

The Robotics Institute
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

December 1991

*Visiting Research Scholar from Southern Illinois University, Carbondale, IL 62901.

TABLE OF CONTENTS

List of Figures

ABSTRACT

Robotic manipulators will play a significant role in the maintenance and repair of space stations and satellites, and other future space missions. Robot path planning and control for the above applications should be optimum, since any inefficiency in the planning may considerably risk the success of the space mission. This paper presents a global optimum path planning scheme for redundant space robotic manipulators to be used in such missions. In this formulation, a variational approach is used to minimize the objective functional. Two optimum path planning problems are considered: first, given the end-effector trajectory, find the optimum trajectories of the joints, and second, given the terminal conditions of the end-effector, find the optimum trajectories for the end-effector and the joints. It is explicitly assumed that the gravity is zero , and the robotic manipulator is mounted on a completely free-flying base (spacecraft) and the attitude control (reaction wheels or thrust jets) is off. Linear and angular momentum conditions for this system lead to a set of mixed holonomic and nonholonomic constraints. These equations are adjoined to the objective functional using a Lagrange multiplier technique. The formulation leads to a system of Differential and Algebraic Equations (DAEs) and a set of terminal conditions. A numerical scheme is presented for forward integration of the above system of DAEs, and an iterative shooting method is used to satisfy the terminal conditions. This approach is significant since most space robots that have been developed so far are redundant. The kinematic redundancy of space robots offers efficient control and provides the necessary dexterity for extra-vehicular activity or avoidance of potential obstacles in space stations.

Acknowledgments

The first author would like to thank Mr. Purushothaman Balakumar of The Robotics Institute for writing a computer program for animation purposes. This program was of considerable value in analyzing the simulation results. The first author would also like to thank Prof. Richard G. Luthy of Civil Engineering for providing office space during his stay at Carnegie Mellon University.

1. Introduction

Space exploration is a new frontier in current science and engineering [1]. Benefits from the space exploration are enormous, however, the stake is also very high. Space missions are hazardous to astronauts [2] because of extremes of temperature and glare, and possible high level of radiation. The extra-vehicular activity also consumes considerable time and may need the dexterity and high load handling capacity that astronauts can not provide. Therefore, using robots in space is beneficial for extending on-orbit time of space shuttle and increasing productivity of space mission.

The use of robotic manipulators in space applications introduces several new problems which do not arise in ground base robot applications. For example, the motion of a space manipulator can cause the base (satellite or spacecraft) of the manipulator to move and disturb the trajectory of the spacecraft [1-4]. This can severely affect the spacecraft performance specially when the mass and the moment of inertia of the manipulator arms and the payloads are not negligible in comparison to the manipulator base.

One solution to the above problem is to use reaction (thrust) jets to control the attitude of the spacecraft (or the robot base) [5-6]. To meet this requirement, the spacecraft must carry additional reaction jet fuel. Since a spacecraft can carry only a limited load, this approach may force removal of other facilities of considerable importance for the success of the mission. Thus it may alter (reduce) the goal of the mission. Furthermore, exhaust from the reaction jets may interfere with proper operation of the instruments on the board. For example, the exhaust may reduce the vision distance of a camera if the camera must see along the reaction jets, and in some extreme cases the deposition of the exhaust on the camera lenses may completely block the vision of the camera.

Another solution to the above problem is to keep the reaction jets off and to move the space manipulator arms in such a fashion that it accomplishes the desired space tasks and yet maintains the stability and the overall trajectory of the spacecraft and/or the satellite. This approach will considerably reduce the reaction jet fuel needed for attitude control and increase the life span of the spacecraft and the manipulator.

It is clear that the second approach is far superior than the first. For this reason, most investigators in this area have focussed their research interests on the second approach [1-10]. Lindberg, Longman, and Zedd [1] address various issues related to free-flying space manipulators and provide a comprehensive review of several papers on the subject. Umetani and Yoshida [7] present a Jacobian matrix formulation for the study of kinematics and control of a free-flying space manipulator. Their formulation includes both the linear and the angular momentum conservation conditions. It should be noted that the pseudo Jacobian inverse solution of the velocity constraint equations provides only local minimum of the generalized velocity norm which may not be the global minimum.

Recently, Vafa and Dubowsky [2] have presented the concept of a Virtual Manipulator (VM) for space robots and have shown that a space robot and the corresponding virtual manipulator give identical kinematic response. They have also shown that if the net linear momentum of the space robotic system is zero, then the location of the virtual ground remains fixed. Thus, using the virtual manipulator concept, the studies and results for ground based manipulators may be extended to space manipulator systems. Papadopoulos and Dubowsky [8] have used this concept to study the singularity of space manipulators.

Longman, Lindberg, and Zedd [3] present a reaction wheel compensation method for attitude control of the spacecraft (or the space manipulator base). Since the reaction wheels use photo-voltaic energy, this method uses considerably less control jet fuel than a reaction jet control method. Reaction wheels, however, do not control the translational disturbances of the spacecraft. Reference [9] provides the kinematics and workspace analyses of a satellite-mounted robotic system.

References [1-9] largely discuss techniques that accomplish certain tasks without disturbing the stability of the systems. Fernandes, Gurvits, and Li [10] present a method for near optimum attitude control of space manipulator using internal motion. This formulation considers a two point boundary value problem but it does not consider the problem of the end-effector following a path. As a result, the holonomic and nonholonomic momentum conservation constraints are homogeneous and the solution for the velocity equations lie in the null space of the Jacobian matrix. This may not be true i' the end-effector must follow certain trajectory. For further discussions, readers may wish to examine references [1-10] and the references therein.

In this paper, we present a global optimum path planning scheme for a redundant space robotic manipulator flying-freely in a zero gravity space with reaction jets off. *Global* here implies that the formulation accounts for minimization of the functional for the entire path and not for the local point. The extra degrees of freedom of the redundant manipulator enlarge the workspace of the manipulator. Furthermore, these extra degrees of freedom may be used to optimize certain functionals and avoid singularities. The formulation accounts for the holonomic and nonholonomic constraint conditions arising from the momentum conservation conditions. We consider the following two optimum path planning problems: first, given the end-effector trajectory, find the optimum trajectories of the joints, and second, given the terminal conditions of the end-effector, find the optimum trajectories for the end-effector and the joints. The formulation leads to a set of differential and algebraic equations. A numerical scheme to solve these set of equations is also presented.

The outline of the paper is as follows: In section two, we develop the momentum conservation and the kinematic conditions for the system. This leads to a set of holonomic and nonholonomic constraints. In this section, we also make some remarks on local optimization and show that in this formulation the second problem may be considered as a subset of the first formulation. In section three, we use a variational approach to optimize an objective functional subjected to the momentum and the kinematic constraints developed in section 2. Boundary conditions are discussed in section four. In section five, we present a brief outline of the iterative shooting method. Section six presents a numerical scheme to integrate the system of differential and algebraic equations. In section seven, we consider a numerical example to show the feasibility of the formulation. Conclusions are presented in section six. Finally, in the Appendix we demonstrate the various matrices that appear in the formulation using a two dimensional example.

2. Momentum Conservation and Kinematic Conditions

In the mathematical formulation for global optimum path planning for space robotic manipulators to follow, we consider the following two problems:

1. Given the trajectory of the end effector, find the trajectories of the joints that minimize a given objective functional, and
2. Given the terminal conditions of the end effector, find the trajectories of the end effector and the joints that minimize a given objective functional.

We assume that the gravity is zero in space and the thrust jets are off. Thus, there is no external force or torque acting on the system and the system is freely floating in the space. This implies that the linear and angular momentum of the system must be conserved.

To develop momentum conservation and other kinematic conditions, consider a typical space robotic manipulator consisting of n_a arms, one base, and n_w inertia wheels (Figure 1). As shown in Figure 1, the manipulator base is numbered 0, the predecessor arms are numbered as 1, 2, $\cdots$, n_a in increasing order and the inertia wheels are numbered greater than n_a. The joint between arm i and its predecessor arm or the base is called joint i. It is assumed that each joint has only one relative Degree Of Freedom (DOF). Thus, the degrees of freedom, n_d, of a spatial and planner space manipulator are, respectively, $(6 + n_a + n_w)$ and $(3 + n_a + n_w)$. Formulations for a system having joints with relative degrees of freedom more than one may be obtained in a similar fashion.

For simplicity in the discussion to follow, the manipulator base, the manipulator arms, and the reaction wheels are also called bodies. Thus body 0 represents the base, bodies 1 to n_a represent the arms, and bodies $(n_a + 1)$ to $(n_a + n_w)$ represent the inertia wheels. As shown in Figure 1, let $\mathbf{r}_i$ be the position vector of center of

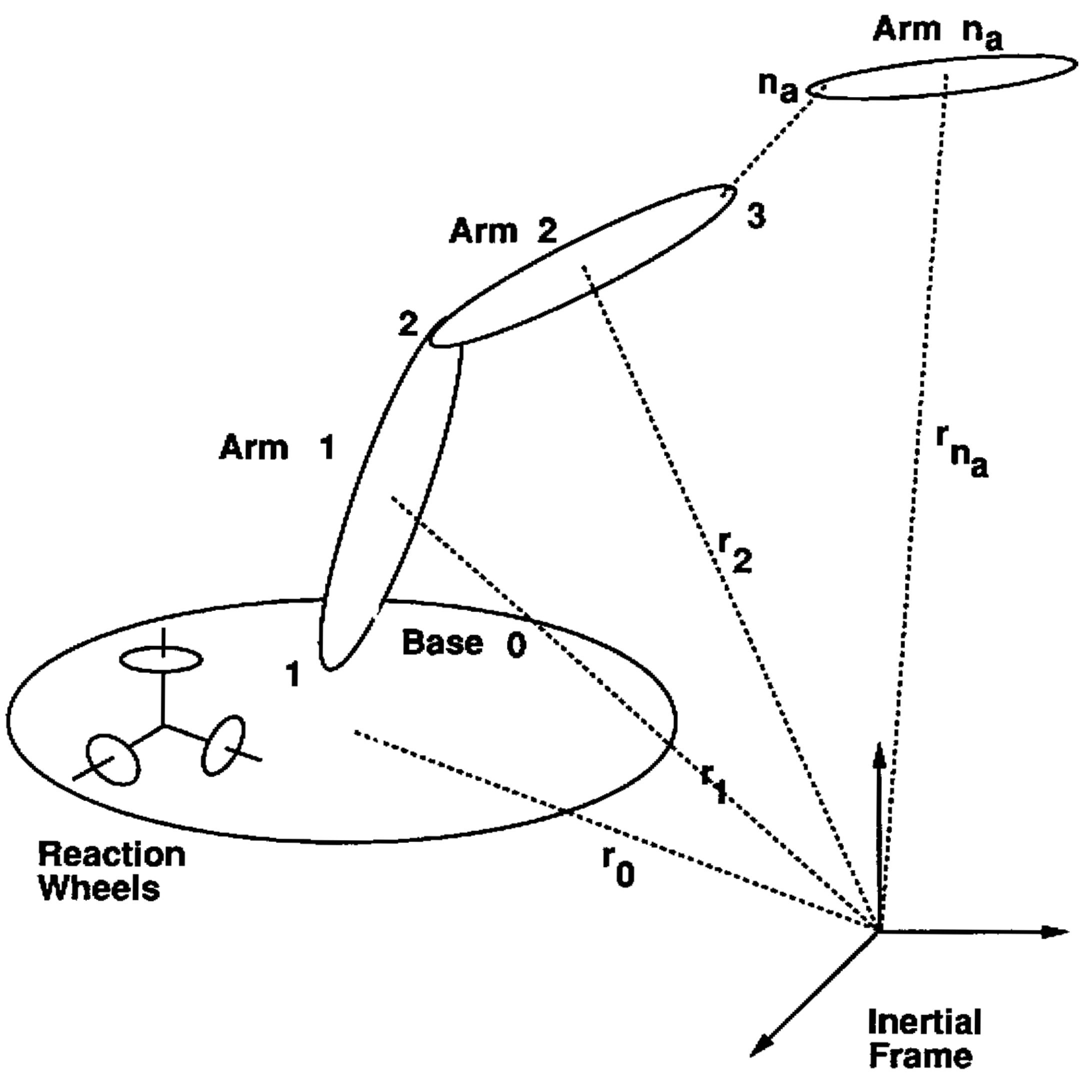

Figure 1. Schematic of a Space Manipulator

mass of body i with respect to (w.r.t.) an inertial frame, then the expressions for the linear and the angular momentum ($\mathbf{p}$ and $\mathbf{L}$) of the system may be written as [11]

$$\sum_{i=0}^{n_a+n_w} m_i \dot{\mathbf{r}}_i = \mathbf{p} \tag{1}$$

$$\sum_{i=0}^{n_a+n_w} [I_i \omega_i + \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i] = \mathbf{L} \tag{2}$$

where m_i and ω_i, are, respectively, the mass and the angular velocity of body i, I_i is the moment of inertia of body i about its center of mass, and the period on a variable $(\dot{.})$ denotes the total derivative of the variable $(.)$ with respect to time t.

In many applications, it is more convenient to include the mass and the moment of inertia of the inertia wheels to that of the manipulator base. Furthermore, for simplicity in the discussion to follow, it is assumed that the linear and the angular momentum of the system are zero, i.e. ($\mathbf{p} = \mathbf{L} = 0$). This is not the limitation of the formulation. Formulations for $\mathbf{p}$ and $\mathbf{L}$ not equal to zero may be developed on a similar line. However, in this case the formulation will be more involved. Following the above discussion, equations (1) and (2) may be written as

$$\sum_{i=1}^{n_a} m_i \dot{\mathbf{r}}_i + m_0' \dot{\mathbf{r}}_0' = \mathbf{0} \tag{3}$$

$$\sum_{i=1}^{n_a} [I_i \omega_i + \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i] + [I_0' \omega_0 + \mathbf{r}_0' \times m_0' \dot{\mathbf{r}}_0'] + \sum_{j=n_a+1}^{n_a+n_w} [I_j \omega_j^{wb}] = \mathbf{0} \tag{4}$$

where m_0' and I_0' are, respectively, mass and moment of inertia of the manipulator base including the inertia properties of the inertia wheels, $\mathbf{r}_0'$ is the new center of mass of the manipulator base, and ω_j^{wb}, $(j = n_a + 1, \cdots, n_a + n_w)$ are the angular velocity of the reaction wheels with respect to the manipulator base.

Equation (4) shows that the attitude of the manipulator base may be controlled in a relatively simple manner by controlling a set of three orthogonally placed inertia wheels. Controlling the attitude of the base without using inertia wheels, however, is more difficult.

For a *general system* one can write a close form solution for the time integral of equation (3) but not of equation (4). Thus, equations (3) and (4) provide a set of holonomic (Eq. 3) and nonholonomic (Eq. 4) conditions. The system must maintain these momentum conservation conditions which cause disturbances in the system. The formulation to follow will show that by proper momentum management one can perform the desired task and still satisfy the equations.

Integrating equation (3) with respect to time, we obtain

$$\sum_{i=1}^{n_a} m_i \mathbf{r}_i + m_0' \mathbf{r}_0' = \mathbf{m}\mathbf{r}_G \tag{5}$$

where $\mathbf{m}$ is the total mass and $\mathbf{r}_G$ is the center of mass of the system. Since the time integral of equation (3) must be a constant, it follows from equation (5) that $\mathbf{r}_G$ should also be a constant, i.e. center of mass of the system must remain fixed in space. Equation (5) provides the linear momentum conservation conditions in the configuration space.

Using Figure 1, the position vector of the end-effector is given as

$$\mathbf{r}_o' + \mathbf{l}_o + \sum_{i=1}^{n_a-1} \mathbf{l}_i + \mathbf{l}_p = \mathbf{r}_p(t) \tag{6}$$

where $\mathbf{l}_o$ is the position vector of joint 1 w.r.t. the center of mass of the manipulator base, $\mathbf{l}_i$ is the position vector of joint $i+1$ w.r.t. joint i, $\mathbf{l}_p$ is the position vector of the end-effector w.r.t. joint n_a, and $\mathbf{r}_p(t)$ is the position vector of the end-effector from the origin of the inertial frame. The orientation of the end-effector may be written in a similar manner.

Equations presented so far are in the vector form and therefore their components may be written in any coordinate system. Let $\mathbf{q} = \begin{bmatrix} q_1, & \cdots, & q_{n_d} \end{bmatrix}^T$ be a vector of generalized coordinates defining the configuration of the system. Selection of generalized coordinates is not the major issue in this paper. However, it is worth mentioning that these coordinates must be chosen very carefully; otherwise, it may lead to mathematical singularity, the resulting code may be computationally inefficient, and in some cases it may fail to give the desired solution. In literature, one of the favorite choices for these coordinates has been to use a set of translational and rotational coordinates of the manipulator base, relative rotation (or translation) of an arm with respect to its predecessor body, and orientation of the inertia wheels with respect to the base. Base rotational coordinates may be represented by Euler angles, Euler like angles, a set of three appropriate parameters, or the Euler parameters. If the Euler parameters are used, then one must impose an additional Euler parameter normalization condition.

In terms of $\mathbf{q}$, equations (5) and (6) may be given, respectively, as

$$\phi_1(\mathbf{q}) = 0 \tag{7}$$

$$\phi_3(\mathbf{q}, t) = 0 \tag{8}$$

In equation (8), time t appears explicitly because of the presence of $\mathbf{r}_p(t)$ in equation (6). Note that if the system is not at a mathematical singular point, then the linear and angular velocity terms are linearly related to the time derivate of $\mathbf{q}$. Based on this fact, equations (3) and (4) and the time derivative of equation (8) may be written as

$$\mathbf{J}_1(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0} \tag{9}$$

$$\mathbf{J}_2(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0} \tag{10}$$

$$\mathbf{J}_3(\mathbf{q}, t)\dot{\mathbf{q}} = \dot{\mathbf{X}}_3 \tag{11}$$

where $\mathbf{J}_3(\mathbf{q}, t) = (\partial\phi_3)/(\partial\mathrm{q})$ and $\dot{\mathbf{X}}_3 = -(\partial\phi_3)/(\partial t)$. Matrices $\mathbf{J}_1(\mathrm{q})$, $\mathbf{J}_2(\mathrm{q})$, and $\mathbf{J}_3(\mathrm{q}, t)$ are called the Jacobian matrices associated with linear momentum (Eq. 3), angular momentum (Eq. 4), and end-effector kinematic conditions (Eq. 6), respectively. Equations (9) to (11) may be written in a combined form as

$$\mathbf{J}(\mathrm{q}, t)\dot{\mathbf{q}} = \dot{\mathbf{X}} \tag{12}$$

where

$$\mathbf{J}(\mathbf{q},t) = \begin{bmatrix} \mathbf{J}_1(\mathbf{q}) \\ \mathbf{J}_2(\mathbf{q}) \\ \mathbf{J}_3(\mathbf{q},t) \end{bmatrix} \tag{13}$$

is the combined Jacobian matrix of the system of dimension $m_J \times n_d$, and

$$\dot{\mathbf{X}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \dot{\mathbf{X}}_3 \end{bmatrix}. \tag{14}$$

For a redundant system, matrix $\mathbf{J}$ is in general a rectangular with $n_d > m_J$. The difference $(n_d - m_J)$, provides the degree of redundancy at the velocity level. Using the pseudoinverse of the Jacobian matrix, the general solution of equation (12) may be written as [12]

$$\dot{\mathbf{q}} = \mathbf{J}^{\#}\dot{\mathbf{X}} + (\mathbf{I} - \mathbf{J}^{\#}\mathbf{J})\mathbf{u} \tag{15}$$

where $\mathbf{J}^{\#}$ is the pseudoinverse of the Jacobian matrix and $\mathbf{u}$ is an arbitrary unknown vector. The first part of this equation is the particular solution of equation (12). This is also the least square solution of the same equation. The second part of equation (15) is the complementary solution or the solution of the homogeneous equations [12].

From equations (12) and (15), the following two observations may be made: first, by setting $\mathbf{u} = 0$ in equation (15) one obtains only a local minimum for $\dot{\mathbf{q}}$ which may or may not give a global minimum [13]. Furthermore, if $\| \dot{\mathbf{X}} \|$ is zero, as is the case in problem two, then equation (15) would lead to a trivial and undesirable solution. Second, if $\dot{\mathbf{X}} \neq 0$, as may be the case in the first problem, then the solution of equation (12) will not lie in the null space of the Jacobian matrix and the second part of equation (15) only will be insufficient to define the solution of equation (12). The present formulation directly works with equation (12) and therefore, it accounts for both the particular and the homogeneous solutions. Note that by eliminating equation (11) from equation (13), one obtains a set of constraints for problem 2. For this reason, the second problem may be considered as a subset of the first problem.

In the next section, we derive the necessary conditions for global optimum path planning for a space robotic manipulator using a variational approach [11].

3. Global Optimization Formulation

A global optimization problem may be stated as follows:

$$\text{Minimize} \qquad I_f = \int_{t_0}^{t_f} f(\mathbf{q}, \dot{\mathbf{q}}, t)dt \tag{16}$$

subjected to the conditions in equation (12). Here I_f is the objective functional, t_0 and t_f are the initial and the final times, and f is some given function. Terminal times t_0 and t_f may be fixed or free. Some of the choices for function f considered in the past are:

$$\text{First} \qquad f(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}}, \tag{17}$$

$$\text{Second} \qquad f(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}\dot{\mathbf{q}}^T A(\mathbf{q}, t)\dot{\mathbf{q}} \tag{18}$$

and

$$\text{Third} \qquad f(\mathbf{q}, \dot{\mathbf{q}}, t) = 1. \tag{19}$$

$f = (1/2)\dot{\mathbf{q}}^T\dot{\mathbf{q}}$ represents minimization of half of the Euclidean velocity norm or the unit-mass-kinetic energy, $f = (1/2)\dot{\mathbf{q}}^T A\dot{\mathbf{q}}$ represents minimization of the weighted Euclidean velocity norm or the kinetic energy if A is the mass matrix, and $f = 1$ represents minimization of time. In many space applications, f may also represent objectives of obstacle avoidance, singularity avoidance, or minimization of the base reaction force.

Here we consider that t_0 and t_f are fixed and function f is given by equation (18). Using the calculus of variations, the variation of equation (16) at the optimum point may be written as

$$\delta I_f = \int_{t_0}^{t_f} \left[(\frac{\partial f}{\partial \mathbf{q}}) - \frac{d}{dt}(\frac{\partial f}{\partial \dot{\mathbf{q}}}) \right] \delta \mathbf{q}(t) dt + \left[\frac{\partial f}{\partial \dot{\mathbf{q}}(t_f)} \right] \delta \mathbf{q}(t_f) - \left[\frac{\partial f}{\partial \dot{\mathbf{q}}(t_0)} \right] \delta \mathbf{q}(t_0) = 0 \quad (20)$$

where $\delta \mathbf{q}(t)$ is the virtual displacement of $\mathbf{q}(t)$. Note that the coefficient of $\delta \mathbf{q}(t)$ in equation (20) in the present form can not be set equal to zero. This is because all components of $\mathbf{q}$ are not independent. For virtual displacement, $\delta \mathbf{q}(t)$ equation (12) leads to

$$\mathbf{J}(\mathbf{q}, t) \delta \mathbf{q} = \mathbf{0} \quad (21)$$

Multiplying equation (21) with λ^T, where λ is the vector of Lagrange multipliers, and integrating the result with respect to time from t_0 to t_f, we obtain

$$\int_{t_0}^{t_f} \lambda^T \mathbf{J}(\mathbf{q}, t) \delta \mathbf{q} dt = \mathbf{0}. \quad (22)$$

Combining equations (20) and (22) we have

$$\int_{t_0}^{t_f} \left[(\frac{\partial f}{\partial \mathbf{q}}) - \frac{d}{dt}(\frac{\partial f}{\partial \dot{\mathbf{q}}}) - \lambda^T \mathbf{J}(\mathbf{q}, t) \right] \delta \mathbf{q}(t) dt + \left[\frac{\partial f}{\partial \dot{\mathbf{q}}(t_f)} \right] \delta \mathbf{q}(t_f) - \left[\frac{\partial f}{\partial \dot{\mathbf{q}}(t_0)} \right] \delta \mathbf{q}(t_0) = 0 \quad (23)$$

Using equation (23) and following the discussion presented in [11], we obtain the following Euler-Lagrange differential equation

$$(\frac{\partial f}{\partial \mathbf{q}})^T - \frac{d}{dt}(\frac{\partial f}{\partial \dot{\mathbf{q}}})^T - \mathbf{J}(\mathbf{q}, t)^T \lambda = 0 \quad (24)$$

and the following terminal conditions

$$\left[\frac{\partial f}{\partial \dot{\mathbf{q}}(t_b)}\right] \delta \mathbf{q}(t_b) = 0 \qquad\qquad t_b = t_0 \text{ or } t_f. \tag{25}$$

Solution of equation (24) which satisfies equations (7), (8), and (12) along with a set of boundary conditions which are consistent with equation (25) yield the desired global optimum joint trajectory. Note that some of the concepts presented here are similar to the approach presented by Kazerounian and Wang [13]. Their formulation, however, does not account for the nonholonomic constraints.

The above formulation is general and applicable to a large class of function f. In particular, for f given by equation (18), equations (24) and (25) reduce to

$$A(\mathbf{q},t)\ddot{\mathbf{q}} + \dot{A}(\mathbf{q},t)\dot{\mathbf{q}} - \frac{1}{2}\left[\frac{\partial (A\dot{\mathbf{q}})}{\partial \mathbf{q}}\right]^T \dot{\mathbf{q}} + \mathbf{J}(\mathbf{q},t)^T \lambda = 0, \tag{26}$$

and

$$\dot{\mathbf{q}}^T(t_b) A(\mathbf{q}(t_0),t) \delta \mathbf{q}(t_b) = 0 \qquad\qquad t_b = t_0 \text{ or } t_f \tag{27}$$

where it is assumed that A is a symmetric matrix. If not, then equation (18) can be modified such that the resulting matrix A is symmetric. A special case is when A is the identity matrix. This is equivalent to considering equation (17) for f. In this case, equations (26) and (27) reduce to

$$\ddot{\mathbf{q}} + \mathbf{J}(\mathbf{q},t)^T \lambda = 0, \tag{28}$$

and

$$\dot{\mathbf{q}}^T(t_b) \delta \mathbf{q}(t_b) = 0 \qquad\qquad t_b = t_0 \text{ or } t_f \tag{29}$$

Equation (26) (or equation (28)) is a set of n_d second order differential equations. This set is equivalent to $2n_d$ first order differential equations. Alternatively, one can use Pontryagin's maximum principle, which maximizes a certain Hamiltonian function, to obtain a similar set of $2n_d$ first order equations. The calculus of variations used here results in simpler equations which are suitable for physical interpretations as well as numerical and symbolic manipulations [13].

4. Boundary Conditions

Equation (26) provides the necessary conditions for functional I_f in equation (16) to be optimum. Kazerounian and Wang [13] present four different sets of boundary conditions for a similar optimization problem. Although, the results of reference [13] are applicable in this formulation also, they must be used with proper understanding. For the sake of completeness, we include the results of reference [13] and outline the essential differences.

In order to obtain a proper set of boundary conditions, we return to equation (27) (Take equation (29) if $A = I$). Note that the generalized coordinates are not all independent. This implies that the virtual displacements are also not all independent, i.e. all components of $\delta \mathbf{q}(t_b)$, $t_b = t_0$ or t_f,in equation (27) (or (29)) cannot be varied freely and they must satisfy equation (21) at the end points. Discussion of reference [13] cannot be applied here in a straight forward manner because of the presence of nonholonomic constraints.

The necessary natural conditions may be evaluated as follows: First, evaluate equation (21) for $t = t_0$, second, multiply the result with μ^T, where μ is the vector of Lagrange multipliers of dimension $m_J \times 1$, third, subtract the result from equation (27) for $t_b = t_0$, and finally repeat the procedure for $t = t_f$. This leads to

$$\left[\dot{\mathbf{q}}^T(t_b)A(\mathbf{q}(t_b),t) - \mu^T(t_b)\mathbf{J}(\mathbf{q}(t_b),t_b)\right]\delta\mathbf{q}(t_b) = 0 \qquad t_b = t_0 \text{ or } t_f \qquad (30)$$

Some remarks on μ will be made in the next section. In the discussion to follow, consider that no generalized coordinate is specified at the end points. In this case the coefficient of $\delta\mathbf{q}(t_b)$, $(t_b = t_0, t_f)$ in equation (30), can be set equal to zero for a proper vector μ and a proper set of independent virtual displacement (see reference [11]). Therefore, for this vector μ and the independent virtual displacements, equation (30) after some algebra, leads to

$$\dot{\mathbf{q}}(t_b) = A^{-1}(\mathbf{q}(t_b), t_b)\mathbf{J}^T(\mathbf{q}(t_b), t_b)\mu(t_b) \qquad t_b = t_0 \text{ or } t_f \tag{31}$$

Equation (12) should be satisfied at time t_b also. This implies

$$\mathbf{J}(\mathbf{q}(t_b), t_b)\dot{\mathbf{q}}(t_b) = \dot{\mathbf{X}}(t_b) \tag{32}$$

From equations (31) and (32), we obtain

$$\mu(t_b) = [\mathbf{J}A^{-1}\mathbf{J}^T]^{-1}\dot{\mathbf{X}}(t_b) \tag{33}$$

Substituting the value of μ back in equation (31), we obtain the generalized velocity vector at time t_b as

$$\dot{\mathbf{q}}(t_b) = A^{-1}\mathbf{J}^T[\mathbf{J}A^{-1}\mathbf{J}^T]^{-1}\dot{\mathbf{X}}(t_b) \qquad t_b = t_0 \text{ or } t_f \tag{34}$$

Equation (34) is referred to as the "natural" boundary conditions. Taking $A = (1/2)I$ in equation (34) we get the same set of natural boundary conditions as that of reference [13].

We are now in a position to obtain the proper boundary conditions. Traditionally, equation (30) is used to obtain the following four sets of boundary conditions:

Set 1. Generalized coordinates are free at both ends: In this case the optimum trajectory must satisfy the natural boundary conditions given by equation (34) at both end points.

Set 2. Generalized coordinates are given at $t = t_0$ but not at $t = t_f$: In this case $\delta\mathbf{q}(t_0) = 0$ and therefore equation (30) for $t_b = t_0$ is automatically satisfied and the optimum trajectory must meet the natural conditions at $t = t_f$ only.

Set 3. Generalized coordinates are given at $t = t_f$ but not at $t = t_0$: This set is identical to set 2. except that the roles of t_0 and t_f have interchanged. In this case $\delta\mathbf{q}(t_f) = 0$ and equation (30) for $t_b = t_f$ is automatically fulfilled and the optimum trajectory needs to agree with the natural conditions at $t = t_0$ only.

Set 4. *Generalized coordinates are specified at both ends: In this* $\delta\mathbf{q}(t_0) = \delta\mathbf{q}(t_f) = 0$ and equation (30) is automatically satisfied for both $t_b = t_0$ and $t_b = t_f$.

Sets 1, 2, and 4 correspond, respectively, to cases 1, 2, and 4, of reference [13]. In the above four sets, it is assumed that at each end either all (independent) generalized coordinates are prescribed or all of them (independent generalized coordinates) are free. This might not be the case. In reality, part of the (independent) generalized coordinates may be prescribed and part of them may be free. Therefore, there are many more possible combination sets which will also satisfy the optimality criteria. More specifically, if a holonomic system has n_d degrees of freedom, then there are 2^{2n_d} possible number of sets which will satisfy the optimality criteria. For a nonholonomic system, as is the case here, this number depends on the number of holonomic and nonholonomic constraints in a complex way.

At this stage, it is worth emphasizing the following point: If a system has n generalized coordinates, then the possible number of sets of independent coordinates is ${}^{n}C_{n_d}$, where C is a combination symbol. These sets correspond to only one set in the optimality criteria discussed above because, for a given set of independent coordinates, the other coordinates are all fixed. Therefore, a given set of independent coordinates can be uniquely mapped to another set of independent coordinates.

Among all possible sets, set 1 stated above gives the minimum value for the object function, because this set imposes no restriction on the system. In practice, it is also possible to have a set of boundary conditions which do not satisfy the optimality criteria. One such set is when the independent generalized coordinates and velocities are specified at time $t = t_0$. This corresponds to case 3 of reference [13]. This set results in an initial value problem. Since, in this case, the natural boundary conditions at $t = t_f$ are ignored, the resulting path is a "weak" minimum. A strategy for obtaining strong minimum for this case is given in [13].

Note that the equations satisfying the optimality criteria lead to split boundary conditions, i.e. two point boundary value problems. A close form solution of these equations is generally not possible. Furthermore, these equations can not be solved in a straight forward manner. A common approach to this problem is as follows: 1) estimate the initial conditions, 2) numerically integrate the differential and algebraic equations, 3) use the numerical results at $t = t_f$ to update the initial conditions, and 4) repeat the steps 2 and 3 until the terminal conditions are satisfied or the number of iteration exceeds a prespecified value. A numerical scheme to integrate the differential and algebraic equations appearing in this formulation is given next.

5. Numerical Integration of Differential and Algebraic Equations

Global optimum path planning formulation presented above leads to a system of Differential and Algebraic Equations (DAEs). Although, the formulation leads to a two point boundary value problem, in this section we begin with some assumed initial conditions, i.e. we take some suitable values for $\mathbf{q}(t_0)$, and $\dot{\mathbf{q}}(t_0)$. A method to improve the initial conditions so that the resulting solution satisfies the boundary conditions at both ends is given in the next section. The objective of this section is to present a numerical scheme to advance the solution of the DAEs from a time grid point t_i to the next time grid point t_{i+1}. For a systematic development and ease of reference, the DAEs are rewritten below:

Differential Equation (Eq. (26)):

$$A(\mathbf{q},t)\ddot{\mathbf{q}} + \mathbf{J}(\mathbf{q},t)^T\lambda = F(\mathbf{q},\dot{\mathbf{q}},t) \tag{35}$$

Holonomic Constraints:

$$\phi(\mathbf{q},t) = \begin{bmatrix} \phi_1(\mathbf{q}) \\ \phi_3(\mathbf{q},t) \end{bmatrix} = \mathbf{0} \tag{36}$$

Non-holonomic Constraints along with the time derivative of equation (36):

$$\mathbf{J}(\mathbf{q},t)\dot{\mathbf{q}} = \dot{\mathbf{X}} \tag{37}$$

Vector $F(\mathbf{q},\dot{\mathbf{q}},t)$ in equation (35) is given as

$$F(\mathbf{q},\dot{\mathbf{q}},t) = \frac{1}{2}\left[\frac{\partial(A\dot{\mathbf{q}})}{\partial\mathbf{q}}\right]^T\dot{\mathbf{q}} - \dot{A}(\mathbf{q},t)\dot{\mathbf{q}} \tag{38}$$

Note that the Lagrange multipliers λ's are unknown. These multipliers may be eliminated from equation (35) as follows: First, differentiate equation (37) with respect to time. This leads to

$$\mathbf{J}(\mathbf{q},t)\ddot{\mathbf{q}} = \ddot{\mathbf{X}} - \dot{\mathbf{J}}(\mathbf{q},t)\dot{\mathbf{q}} \tag{39}$$

Second, substitute the value of $\ddot{\mathbf{q}}$ from equation (35) into equation (39), and solve for λ. This gives

$$\lambda = \left[JA^{-1}J^T\right]^{-1}(JA^{-1}F - \ddot{\mathbf{X}} + \dot{\mathbf{J}}\dot{\mathbf{q}}) \tag{40}$$

Finally, substitute the expression for λ back into equation (35). After some algebra, this leads to

$$\ddot{\mathbf{q}} = A^{-1}F - A^{-1}J\left[JA^{-1}J^T\right]^{-1}(JA^{-1}F - \ddot{\mathbf{X}} + \dot{\mathbf{J}}\dot{\mathbf{q}}) \tag{41}$$

Before we proceed further, note that λ (equation (40)) at $t = t_0$ (or t_f) is different from $\mu(t_0)$ (or $\mu(t_f)$). In reality, λ and μ are entirely two different vectors and they should be treated so.

The right hand side of equation (41) contains no unknown terms. Therefore, given $\mathbf{q}(t_i)$, $\dot{\mathbf{q}}(t_i)$, and t_i, equation (41) may be used to compute $\ddot{\mathbf{q}}(t_i)$. Another approach to compute $\ddot{\mathbf{q}}(t_i)$ is as follows: Equations (35) and (39) may be written in combined form as

$$\begin{bmatrix} A & \mathbf{J}^T \\ \mathbf{J} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ \ddot{\mathbf{X}} - \dot{\mathbf{J}}\dot{\mathbf{q}} \end{bmatrix} \tag{42}$$

In this equation, the square matrix and the right hand vector can be computed numerically. Vectors $\ddot{\mathbf{c}}$ and λ then can be solved using a numerical scheme such as the Gaussian elimination technique. Thus, vector $\ddot{\mathbf{q}}$ may be obtained using either equation (41) or equation (42). Equation (42) has two major advantages over equation (41). First, equation (42) preserves the sparsity of the matrices and therefore a sparse matrix code may be used to store the matrix elements and solve the resulting equations, and second, if the eigenvalues of $\mathbf{J}$ vary widely then equation (41) results in ill condition matrices which cause numerical instability. For the same Jacobian matrix, equation (42) leads to a relatively well behaved problem.

Let the solution for vector $\ddot{\mathbf{q}}$ be written symbolically as

$$\ddot{\mathbf{q}} = g(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{43}$$

The vector function g is never computed explicitly but only numerically. Equation (43) may be reduced to a set of first order equations as

$$\dot{y} = \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} g(\mathbf{q}, \mathbf{u}, t) \\ \mathbf{u} \end{bmatrix} = g_1(\mathbf{y}, t) \tag{44}$$

Equation (44) may now be integrated using a direct integration scheme. This scheme may lead to numerical problems because the components of $\mathbf{q}$ are not all independent due to the presence of nonlinear holonomic and nonholonomic constraints. These constraints nonlinearly transform the small numerical integration errors which act as feedback to the system causing large constraint violation and numerical instability. Furthermore, since the dimension of $\mathbf{q}$ may be much larger than the number of independent coordinates, these schemes require integration of a much larger set of differential equations.

The DAEs in this formulation are similar to those in multi-body dynamics. In recent years, several methods have appeared in the area of multibody dynamics to solve such DAEs. Some of these methods attempt to satisfy the constraints explicitly and some implicitly, and some take a hybrid approach. A brief review of these methods appear in reference [14]. From theory of differential geometry, it is clear that the solution of these DAEs lies on the manifolds defined by these constraints. It is possible to define coordinate systems on these manifolds and reduce the number of differential equations to its minimum (see reference [14] and the references therein). In this case, the resulting generalized coordinates are no longer the original set but the combination of original coordinates with time varying coefficients. In this paper we shall present the numerical to solve the DAEs in terms of a set of coordinates which is a subset of the original set of generalized coordinates.

In order to discuss the numerical scheme to follow, let $\mathbf{q}_{I1}(t_i)$ and $\dot{\mathbf{q}}_{I2}(t_i)$ be the vectors of independent generalized coordinates and velocities at any time t_i. These two vectors completely define the state of the system, because they may be used to solve the dependent generalized coordinates and velocities. Note that the dimensions of these two vectors are not the same because of the presence of nonholonomic constraints. Once these vectors have been identified, the numerical scheme to solve the DAEs may be given as follows:

Numerical Algorithm:

Step 1. Use $\mathbf{q}_{I1}(t_i)$ and equation (36) to solve the unknown dependent coordinate. This requires solution of a set of nonlinear equations. Newton-Raphson method and its variants may be used for this purpose.

Step 2. Use $\dot{\mathbf{q}}_{I2}(t_i)$, equation (37) and the vector $\mathbf{q}(t_i)$ obtained in step 1 to solve for the dependent velocities. This requires solution of a set of linear equations. Any of the many schemes such as Gaussian elimination, etc. may be used for this purpose.

Step 3. Solve for $\ddot{\mathbf{q}}(t_i)$ using either equation (41) or (42) and identify vector $\ddot{\mathbf{q}}_{I2}(t_i)$ from this vector.

Step 4. Use vectors $\ddot{\mathbf{q}}_{I2}(t_i)$ and $\dot{\mathbf{q}}_{I2}(t_i)$ in an integration subroutine to obtain vectors $\dot{\mathbf{q}}_{I2}(t_{i+1})$ and $\mathbf{q}_{I1}(t_{i+1})$.

Step 5. Repeat steps 1 to 4 until final time has reached.

Steps 1 and 2 insure that the kinematic conditions are satisfied. Also note that the number of equations that needs to be integrated is much less than the number of differential equations in (44). In this approach, however, one must solve additional linear and nonlinear equations.

6. Iterative Shooting Method

A numerical method to solve the DAEs was presented in the previous section. As shown above, the problem addressed here leads to a two-point boundary value problem. The assumed initial guess in the previous section generally does not satisfy the boundary conditions. The guess may be improved using a shooting method. The basic idea of the shooting method is as follows: First, start with an initial guess and find the response at the final time, second, use some numerical scheme to estimate the initial guess, and third, repeat the first and the second steps until the boundary conditions are satisfied or the iteration limit is reached.

In this section we consider the Secant method to improve the initial guess. This is a well established technique and the detailed numerical algorithm for this may be obtained in many of the standard numerical analysis literature. Here, we briefly outline the method. Let the two-point-boundary value problem be as follows: Find the response of the system

$$\ddot{\mathbf{y}} = f(\mathbf{y}, \dot{\mathbf{y}}, t) \tag{45}$$

which satisfies the following boundary conditions: $\mathbf{y}(t_0) = \mathbf{y}_0$ and $\mathbf{y}(t_f) = \mathbf{y}_f$. In this method two initial guesses for $\dot{\mathbf{y}}(t_0)$ are considered. The Secant method for improving the initial guess for $\dot{\mathbf{y}}(t_0) = \dot{\mathbf{y}}_0$ is given as follows:

$$\dot{\mathbf{y}}^k(t_0) = \dot{y}^{k-1}(t_0) - \frac{(\mathbf{y}^{k-1}(t_f) - \mathbf{y}_f)(\dot{\mathbf{y}}^{k-1}(t_0) - \dot{\mathbf{y}}^{k-2}(t_0))}{\mathbf{y}^{k-1}(t_f) - \mathbf{y}^{k-2}(t_f)} \tag{46}$$

where k is the iteration number. This scheme generates a sequence of $\dot{\mathbf{y}}^k(t_0)$ such that

$$\lim_{k \to \infty} \mathbf{y}^k(t_f) = \mathbf{y}_f$$

Thus, the initial guess that satisfies the two-point-boundary conditions may be obtained.

7. Numerical Results

In order to demonstrate the feasibility of the formulation, we consider a two-dimensional space manipulator consisting of a base (body 0) and three arms (bodies 1, 2, and 3) as shown in Figure 2. The link lengths are $L_1 = L_2 = 7.0m$, and $L_3 = 4.0m$. Initially, the inertia properties considered are $m_0 = 100kg.$, $m_1 = m_2 = 7.0kg.$, $m_3 = 4.0kg.$, $I_0 = 1.0(kg - m^2)$, $I_1 = I_2 = 28.583(kg - m^2)$, and $I_3 = 5.333(kg - m^2)$. The tip of the vector is required to move along a circular trajectory defined as

$$\mathbf{X} = \begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} x_c + r cos(\alpha) \\ y_c + r sin(\alpha) \end{bmatrix} \tag{47}$$

where (x_c, y_c) is the center of the circle, r its radius, and α a specified time dependent parameter. The center is considered at $x_c = 11.0m$ and $y_c = 6.0m$, and the radius $r = 3.0m$. $\alpha(t)$ considered is

$$\alpha(t) = \frac{2\pi}{9} - sin(\frac{2\pi t}{9}) \tag{48}$$

The objective is to find an optimal trajectory for a set of specified initial generalized coordinates. Also, the initial and the final velocities should be zero. Furthermore, the robot base is allowed to translate, but not to rotate. The initial conditions, which are consistent with the constraints, are $x_0 = -1.0213$, $y_0 = -0.7124$, $\theta_1 = 0.93478$, $\theta_2 = 0.0064397$, and $\theta_3 = 0.262$, and all initial velocities are zero.

The numerical results of the system are given below. Figure 3 shows the displacements Δx_0 and Δy_0 as a function of time. From this figure, it is clear that the motion of the arms considerably deflects the center of mass of the base. The changes in the orientations of the arms as a function of time are shown in Figure 4. Figure 5 shows configurations of the space manipulator at various times. Numerical results for this response were also studied using an animation program. It is observed that the end-effector traces a perfect circle and the response of the system is smooth and continuous.

In a further investigation, the mass and the inertia of each arm were increased,

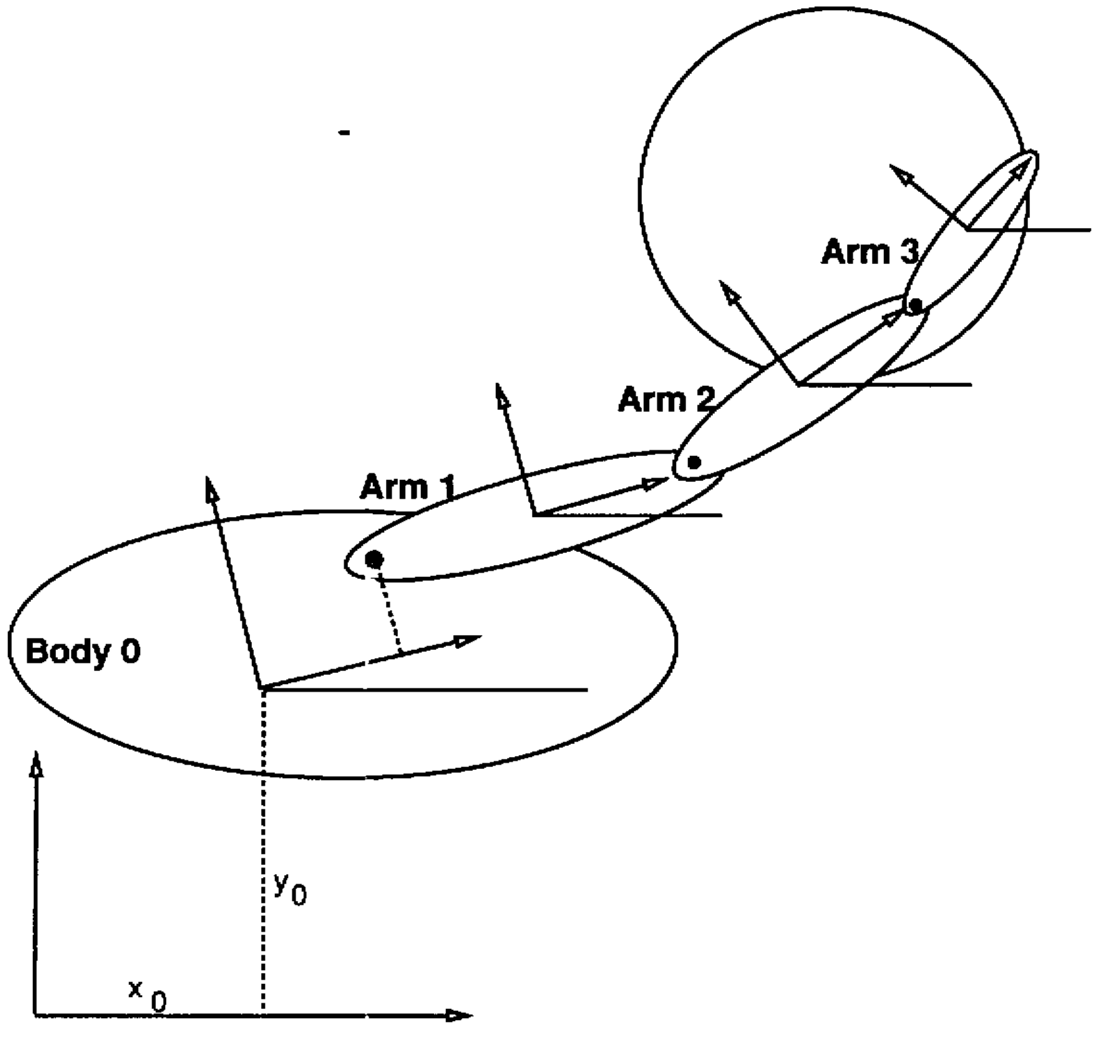

Figure 2. Configuration of a Two Dimensional Space Manipulator

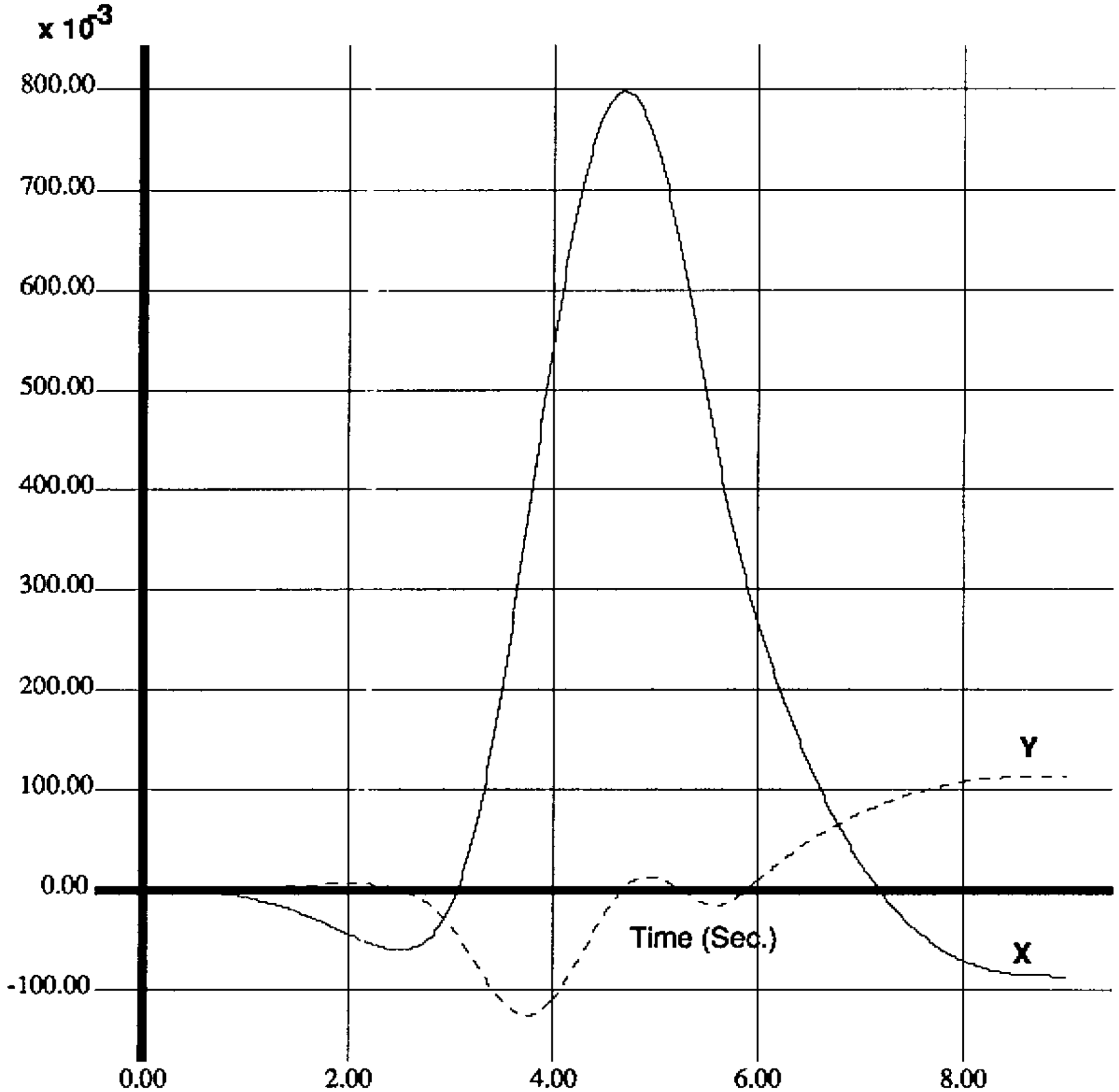

Figure 3. Time Response of the Center of Mass of the Base

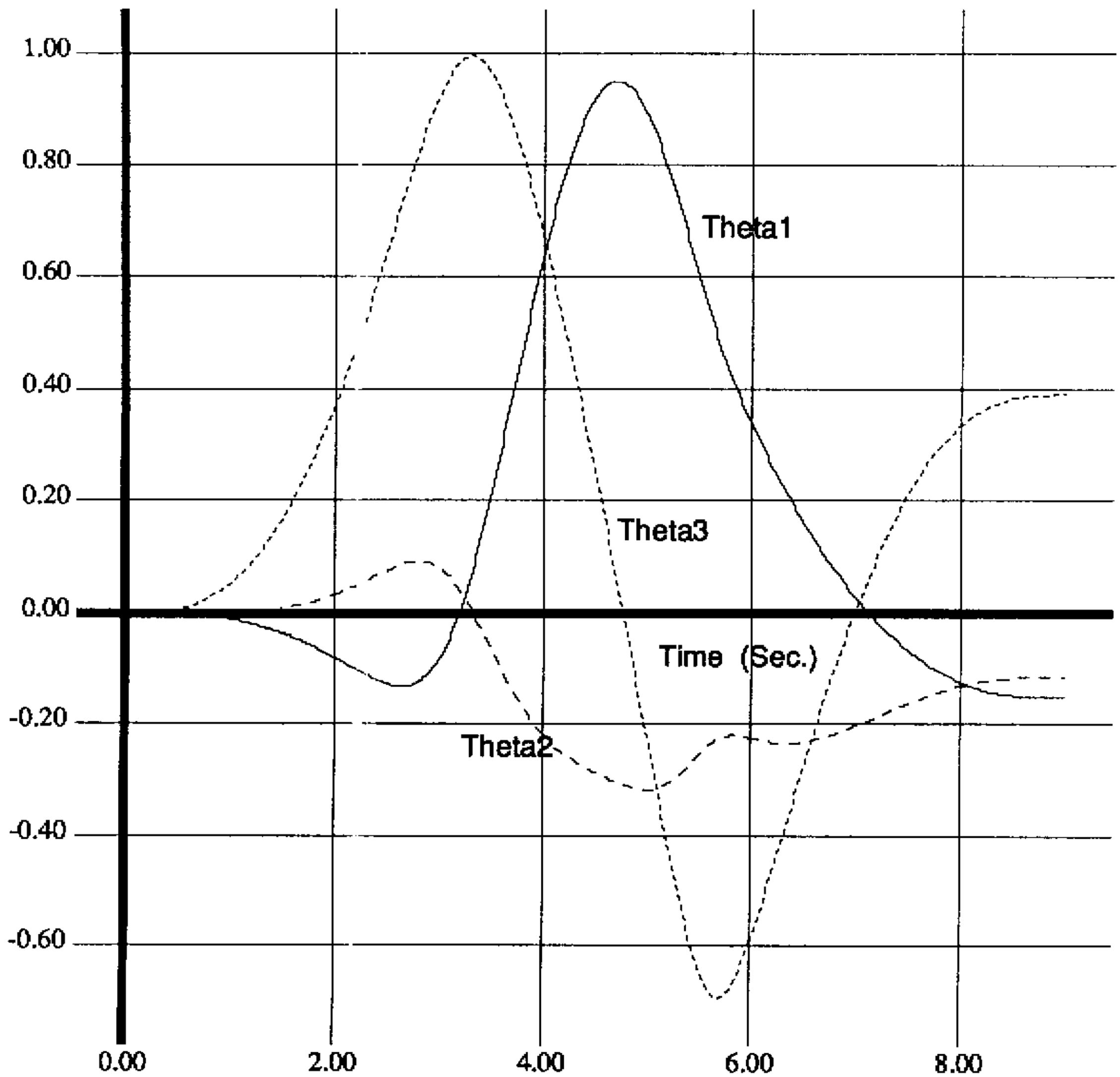

Figure 4. Orientations of the Manipulator Arms as a Function of Time

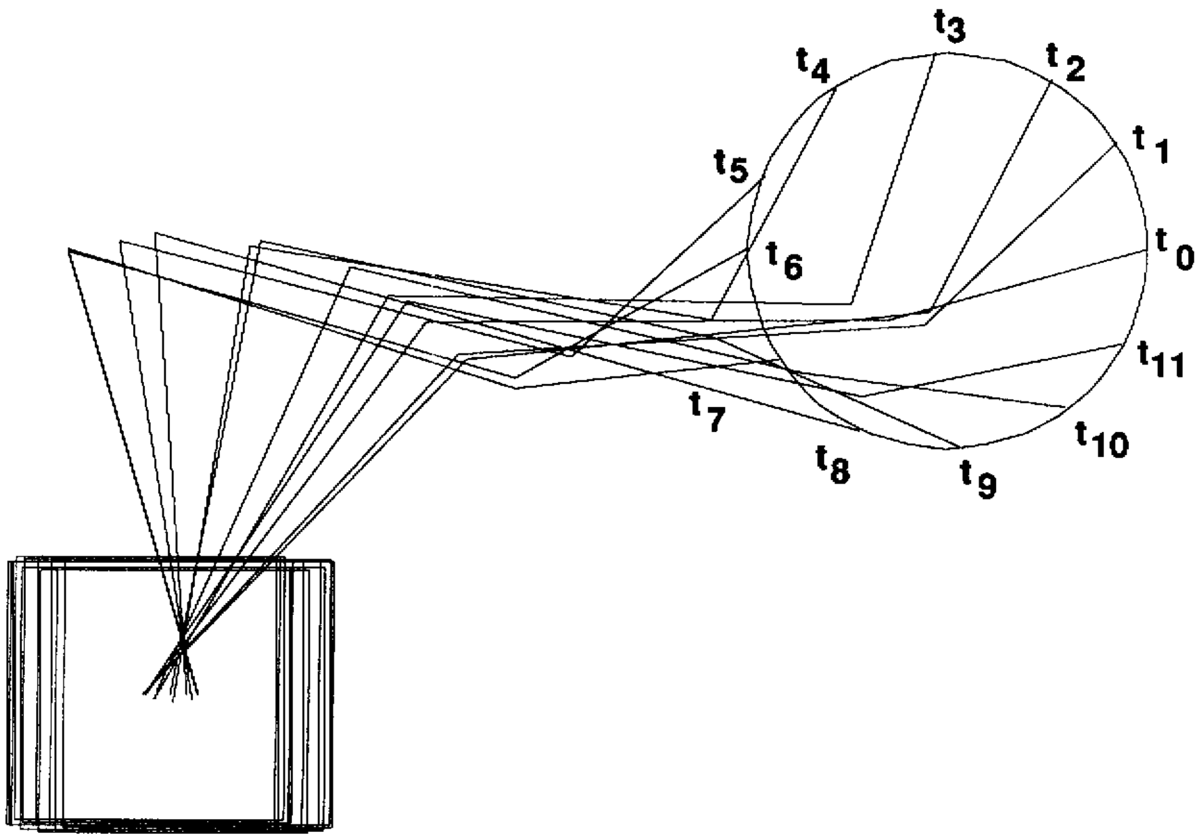

Figure 5. Configuration of the Manipulator at Various Points

first by three times and second by six times. Corresponding changes in time response for Δx_0, Δy_0, $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$ are shown in Figures 6 to 10. It can be seen that the variation in the response of the base increases with increase in inertia. This is in agreement with the intution.

8. Conclusions:

An optimum path planning formulation for a free floating space robotic manipulator based on a variational approach has been presented. The formulation shows that the moment conservation conditions result in a system of holonomic and nonholonomic constraints. A method to incorporate these constraints has also been given. This leads to a system of differential and algebraic equations, and two-point boundary conditions if the initial conditions are not specified. A numerical algorithm to solve the resulting DAEs has been proposed. Also, a numerical scheme to solve the two point boundary value problem has been outlined. The formulation has been used to obtain the response of a two-dimensional redundant space robot. The numerical results show that the motion of the arms can considerably affect the response of the base. The approach is of significance in many space applications since most current space robots are redundant. This redundancy provides the necessary dexterity required for extra-vehicular activity or avoidance of potential obstacles in space stations.

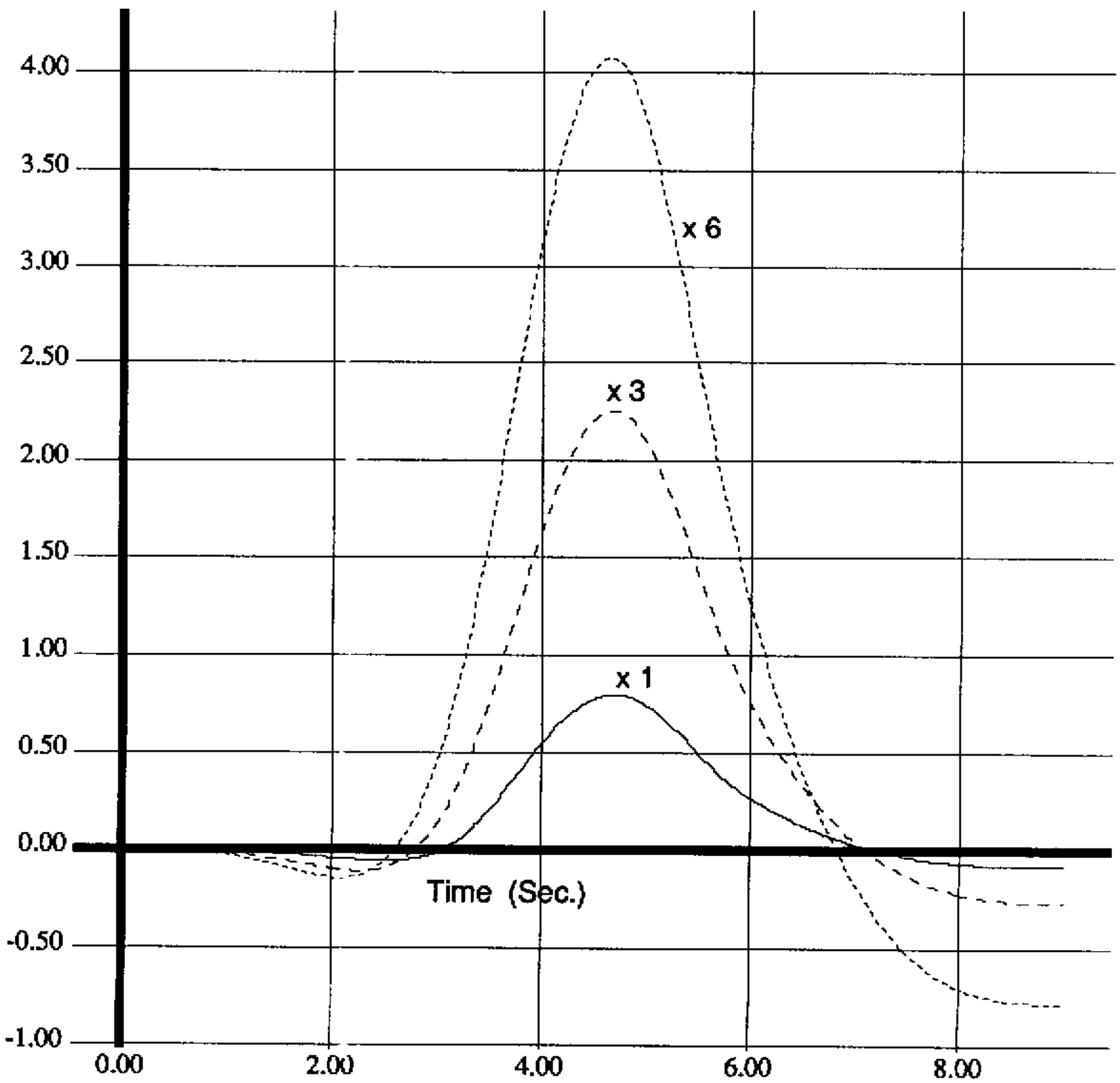

Figure 6 Effect of Change of Mass on $X_0(t)$

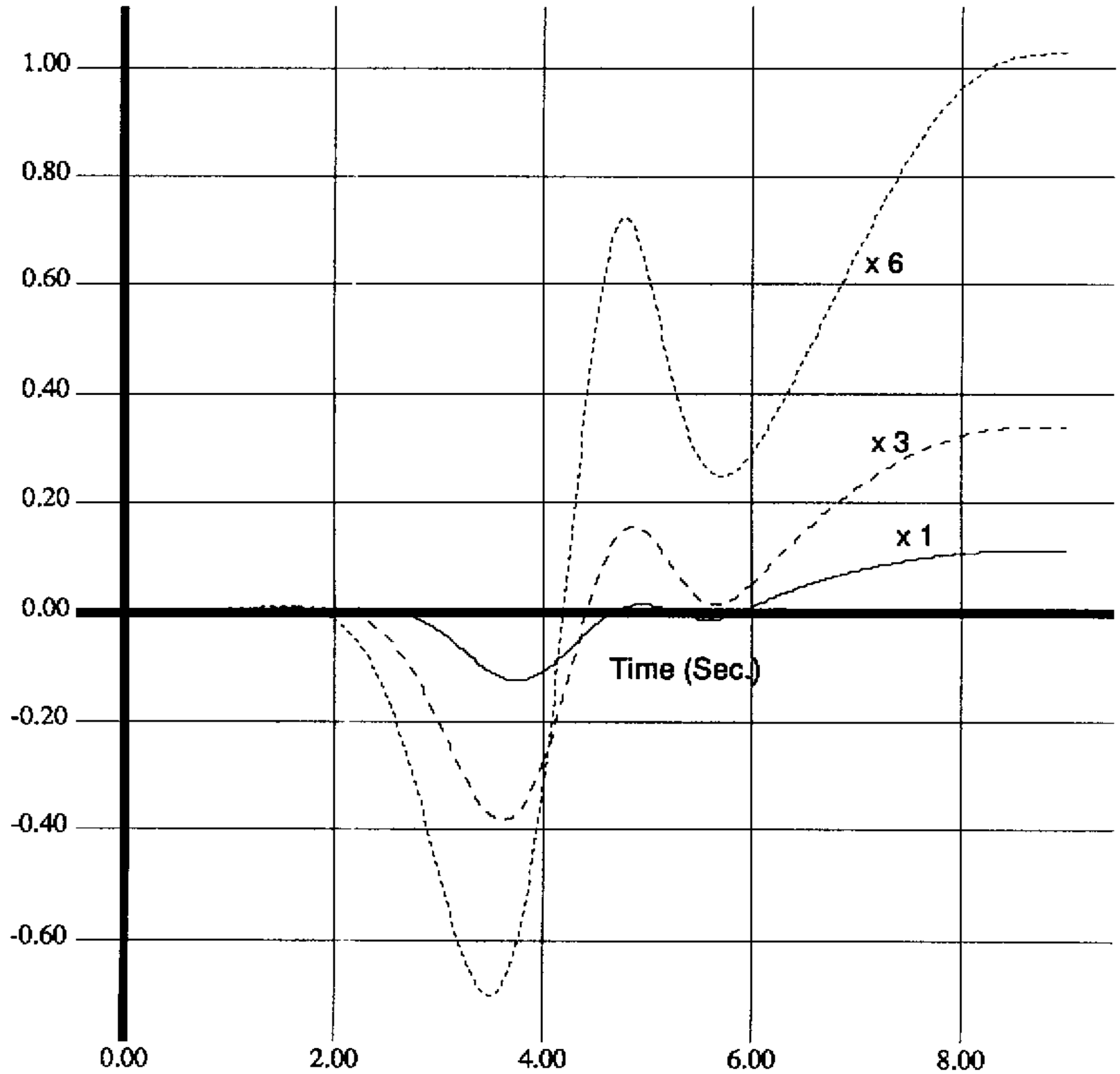

Figure 7. Effect of Change of Mass on $Y_0(t)$

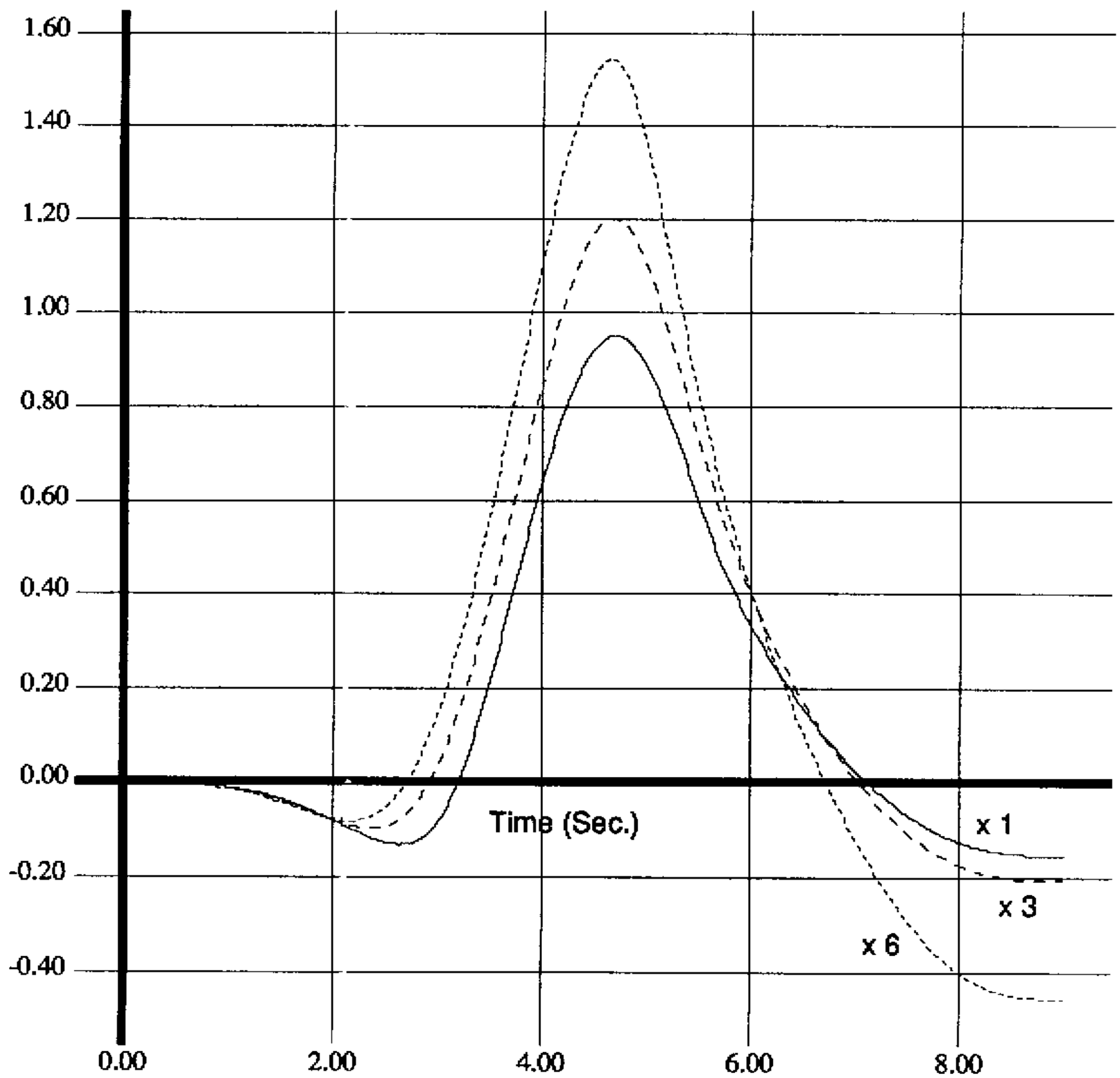

Figure 8. Effect of Change of Mass on theta 1(t)

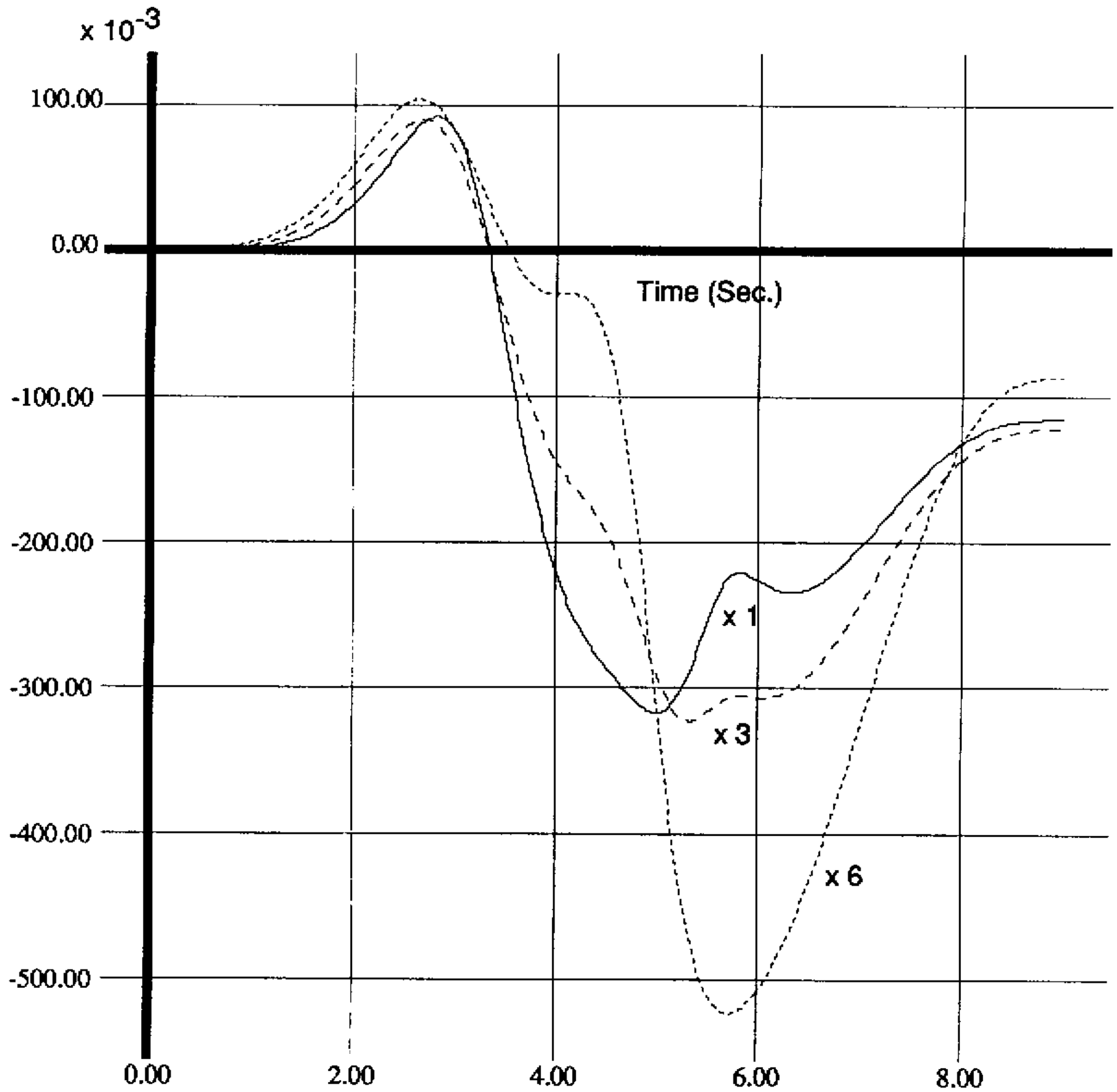

Figure 9. Effect of Change of Mass on theta $_2$(t)

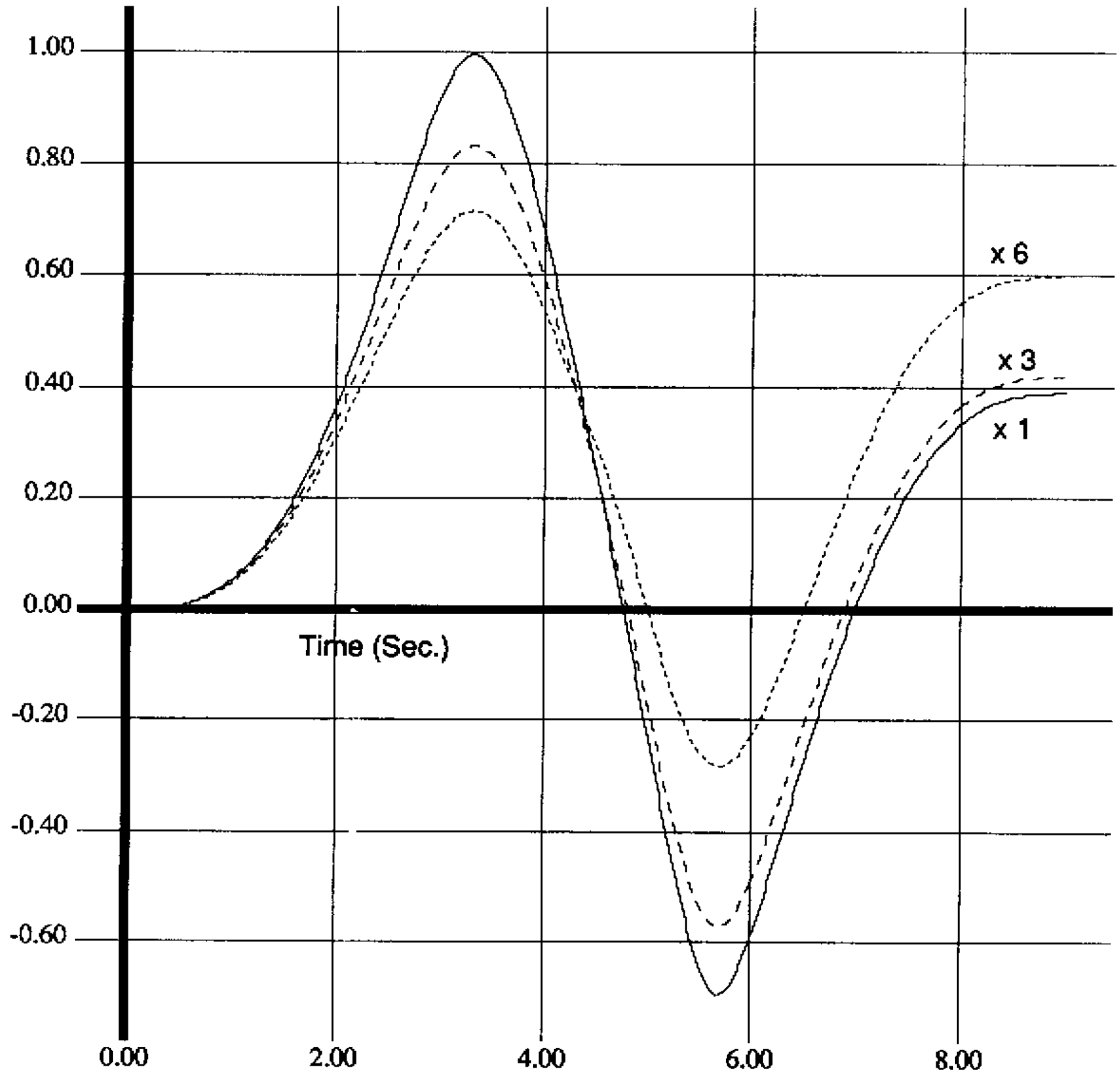

Figure 10. Effect of Change of Mass on theta (t)

APPENDIX

Equations (7) to (11) are the main equations in the derivation for global optimum path planning for a redundant space robotic manipulator. Once these equations are known, the other equations may be obtained using the formulation presented in this paper. In this section we demonstrate these equations for a two dimensional space manipulator system.

In order to accomplish the above stated objective, consider a two dimensional space manipulator consisting of one base (body 0) and three arms (bodies 1, 2, and 3) as shown in figure 2. The configuration of this system is defined using two sets of frames, namely an inertial frame and four body frames. The origin of each body frame is rigidly attached to the center of mass of the respective body. For simplicity, it is assumed that the total linear and angular momentums of the system are zero and the origin of the inertial frame coincides with the center of mass of the overall system. For this system, the vector of generalized coordinates $\mathbf{q}$ taken here is

$$\mathbf{q} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & x_0 & y_0 & \theta_0 \end{bmatrix}^T \tag{A1}$$

where θ_i is the orientation of the x-axis of body frame i with respect to the x-axis of the inertial frame, and (x_0, y_0) is the location of center of mass with respect to the inertial frame. Let m_i and I_i be the mass and moment of inertia of body i and m be the total mass of the system; i.e.

$$m = m_0 + m_1 + m_2 + m_3 \tag{A2}$$

The linear and angular momentum conservation conditions for this system lead to

$$\sum_{i=0}^{3} m_i \dot{\mathbf{r}}_i = 0 \tag{A3}$$

$$\sum_{i=0}^{3} (I_i \theta_i + m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i) = 0 \tag{A4}$$

where $\mathbf{r}_i$ is the position vector of center of mass of body i, and the period on (*) denoted total time derivate of (*). Vectors $\mathbf{r}_i$ $(i = 0, 1, 2, 3)$ may be written in terms of q directly from figure 2. Substituting the expression for $\mathbf{r}_i$ in equations (A3) and (A4), we obtain equations (8) and (9). For this case, the components of matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ are as follows:

$$[\mathbf{J}_1]_{11} = -(\frac{m_1}{2} + m_2 + m_3) * L_1 * sin(\theta_1) \tag{A5}$$

$$[\mathbf{J}_1]_{12} = -(\frac{m_2}{2} + m_3) * L_2 * sin(\theta_2) \tag{A6}$$

$$[\mathbf{J}_1]_{13} = -\frac{m_3}{2} * L_3 * sin(\theta_3) \tag{A7}$$

$$[\mathbf{J}_1]_{14} = m \tag{A8}$$

$$[\mathbf{J}_1]_{15} = 0 \tag{A9}$$

$$[\mathbf{J}_1]_{16} = (m - m_0) * (-\xi * sin(\theta_0) - \eta * cos(\theta_0)) \tag{A10}$$

$$[\mathbf{J}_1]_{21} = (\frac{m_1}{2} + m_2 + m_3) * L_1 * cos(\theta_1) \tag{A11}$$

$$[\mathbf{J}_1]_{22} = (\frac{m_2}{2} + m_3) * L_2 * cos(\theta_2) \tag{A12}$$

$$[\mathbf{J}_1]_{23} = \frac{m_3}{2} * L_3 * cos(\theta_3) \tag{A13}$$

$$[\mathbf{J}_1]_{24} = 0 \tag{A14}$$

$$[\mathbf{J}_1]_{25} = m \tag{A15}$$

$$[\mathbf{J}_1]_{26} = (m - m_0) * (\xi * cos(\theta_0) - \eta * sin(\theta_0)) \tag{A16}$$

$$[\mathbf{J}_2]_{11} = I_1 + (\frac{m_1}{2} + m_2 + m_3)L_1(x_0cos(\theta_1) + y_0sin(\theta_1) + \xi cos(\theta_1 - \theta_0) + \eta sin(\theta_1 - \theta_0))$$

$$+(\frac{m_1}{4} + m_2 + m_3)L_1^2 + (\frac{m_2}{2} + m_3)L_1L_2cos(\theta_1 - \theta_2) + \frac{m_3}{2}L_1L_3cos(\theta_1 - \theta_3) \quad (A17)$$

$$[\mathbf{J}_2]_{12} = I_2 + (\frac{m_2}{2} + m_3)L_2(x_0cos(\theta_2) + y_0sin(\theta_2) + \xi cos(\theta_2 - \theta_0) + \eta sin(\theta_2 - \theta_0))+$$

$$(\frac{m_2}{2} + m_3)L_1L_2cos(\theta_2 - \theta_1) + (\frac{m_2}{4} + m_3)L_2^2 + \frac{m_3}{2}L_2L_3cos(\theta_2 - \theta_3) \quad (A18)$$

$$[\mathbf{J}_2]_{13} = I_3 + \frac{m_3}{2}L_3((x_0cos(\theta_3) + y_0sin(\theta_3) + \xi cos(\theta_3 - \theta_0) + \eta sin(\theta_3 - \theta_0)$$

$$+L_1cos(\theta_3 - \theta_1) + L_2cos(\theta_3 - \theta_2) + \frac{1}{2}L_3) \quad (A19)$$

$$[\mathbf{J}_2]_{14} = -my_0 - (m - m_0)(\xi sin(\theta_0) + \eta cos(\theta_0)) - (\frac{m_1}{2} + m_2 + m_3)L_1sin(\theta_1)$$

$$-(\frac{m_2}{2} + m_3)L_2sin(\theta_2) - \frac{m_3}{2}L_3sin(\theta_3) \quad (A20)$$

$$[\mathbf{J}_2]_{15} = mx_0 + (m - m_0)(\xi cos(\theta_0) - \eta sin(\theta_0)) + (\frac{m_1}{2} + m_2 + m_3)L_1cos(\theta_1)$$

$$+(\frac{m_2}{2} + m_3)L_2cos(\theta_2) + \frac{m_3}{2}L_3cos(\theta_3) \quad (A21)$$

$$[\mathbf{J}_2]_{16} = I_0 + (m-m_0)(\xi^2+\eta^2) + ((m-m_0)x_0 + (\frac{m_1}{2}+m_2+m_3)L_1cos(\theta_1) + (\frac{m_2}{2}+m_3)L_2cos(\theta_2)$$

$$+\frac{m_3}{2}L_3cos(\theta_3))(\xi cos(\theta_0) - \eta sin(\theta_0)) + ((m - m_0)y_0 + (\frac{m_1}{2} + m_2 + m_3)L_1sin(\theta_1)+$$

$$(\frac{m_2}{2} + m_3)L_2sin(\theta_2) + \frac{m_3}{2}L_3sin(\theta_3))(\xi sin(\theta_0) + \eta cos(\theta_0)) \quad (A22)$$

Equations (A20) and (A21) follow from linear momentum conservation conditions. Integrating equation (A3) and using the fact that the center of mass of the system lies at (0,0), we obtain equation (7) as

$$\phi_1(\mathbf{q}) = \begin{bmatrix} \phi_{11}(\mathbf{q}) \\ \phi_{12}(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A23)$$

where functions $\phi_{11}(\mathbf{q})$ and $\phi_{12}(\mathbf{q})$ are given as

$$\phi_{11}(\mathbf{q}) = mx_0 + (m - m_0)(\xi cos(\theta_0) - \eta sin(\theta_0)) + (\frac{m_1}{2} + m_2 + m_3)L_1 cos(\theta_1) +$$
$$(\frac{m_2}{2} + m_3)L_2 cos(\theta_2) + \frac{m_3}{2} L_3 cos(\theta_3) = 0 \quad (A24)$$

$$\phi_{12}(\mathbf{q}) = my_0 + (m - m_0)(\xi sin(\theta_0) + \eta cos(\theta_0)) + (\frac{m_1}{2} + m_2 + m_3)L_1 sin(\theta_1) +$$
$$(\frac{m_2}{2} + m_3)L_2 sin(\theta_2) + \frac{m_3}{2} L_3 sin(\theta_3) = 0 \quad (A25)$$

Equations (A24) and (A25) lead to $[\mathbf{J}_2]_{14} = [\mathbf{J}_2]_{14} = 0$. Let $\mathbf{X}_3 = \begin{bmatrix} x_p(t) & y_p(t) \end{bmatrix}^T$ be the specified trajectory of the end effector. Then the trajectory constraint (equations (8)) is given as

$$\phi_3(\mathbf{q}, t) = \begin{bmatrix} \phi_{31}(\mathbf{q}, t) \\ \phi_{32}(\mathbf{q}, t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A26)$$

where functions $\phi_{31}(\mathbf{q}, t)$ and $\phi_{32}(\mathbf{q}, t)$ are given by the following equations:

$$\phi_{31}(\mathbf{q}, t) = x_0 + \xi * cos(\theta_0) - \eta * sin(\theta_0) + L_1 * cos(\theta_1) + L_2 * cos(\theta_2) + L_3 * cos(\theta_3) - x_p(t) = 0 \quad (A27)$$

$$\phi_{32}(\mathbf{q}, t) = y_0 + \xi * sin(\theta_0) + \eta * cos(\theta_0) + L_1 * sin(\theta_1) + L_2 * sin(\theta_2) + L_3 * sin(\theta_3) - y_p(t) \quad (A28)$$

In this study, the trajectory is a circle specified as

$$\mathbf{X}_3 = \begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} x_c + rcos(\alpha(t)) & y_c + rsin(\alpha(t)) \end{bmatrix} \quad (A29)$$

where (x_c, y_c) and r are, respectively, the coordinate of the center and the radius of the circle, and $\alpha(t)$ is a specified time dependent parameter. In this study, $\alpha(t)$ is considered as

$$\alpha(t) = \frac{2\pi}{9} - sin(\frac{2\pi t}{9}) \qquad (A30)$$

Time derivative of equations (A27) and (A28) gives equation (11). The components of the Jacobian matrix $\mathbf{J}_3$ are as follows:

$$[\mathbf{J}_3]_{11} = -L_1 * sin(\theta_1) \qquad (A31)$$

$$[\mathbf{J}_3]_{12} = -L_2 * sin(\theta_2) \qquad (A32)$$

$$[\mathbf{J}_3]_{13} = -L_3 * sin(\theta_3) \qquad (A33)$$

$$[\mathbf{J}_3]_{14} = 1 \qquad (A34)$$

$$[\mathbf{J}_3]_{15} = 0 \qquad (A35)$$

$$[\mathbf{J}_3]_{16} = -\xi * sin(\theta_0) - \eta * cos(\theta_0) \qquad (A36)$$

$$[\mathbf{J}_3]_{21} = L_1 * cos(\theta_1) \qquad (A37)$$

$$[\mathbf{J}_3]_{22} = L_2 * cos(\theta_2) \qquad (A38)$$

$$[\mathbf{J}_3]_{23} = L_3 * cos(\theta_3) \qquad (A39)$$

$$[\mathbf{J}_3]_{24} = 0 \qquad (A40)$$

$$[\mathbf{J}_3]_{25} = 1 \tag{A41}$$

$$[\mathbf{J}_3]_{26} = \xi * cos(\theta_0) - \eta * sin(\theta_0) \tag{A42}$$

and the vector $\dot{\mathbf{X}}_3$ is given as

$$\dot{\mathbf{X}}_3 = \frac{2\pi}{9}(1 - cos(\frac{2\pi t}{9})) \begin{bmatrix} -r sin(\alpha) \\ r cos(\alpha) \end{bmatrix} \tag{A43}$$

Thus, equations (7) to (9) are known for the system.

References

1. R.E. Lindberg, R.W. Longman and M.F. Zedd, "Kinematics and Dynamic Properties of an Elbow Manipulator Mounted on a Satellite," *The Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 397-421.

2. Z. Vafa and S. Dubowsky, "On the Dynamics of Space Manipulators Using the Virtual Manipulator Approach with Applications to Path Planning," *The Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 441-472.

3. R.W. Longman, R.E. Lindberg and M.F. Zedd, "Satellite Mounted Robot Manipulators-New Kinematics and Reaction Moment Compensator," *International Journal of Robotic Research*, Vol. 6, No. 3, Fall 1987, pp. 87-103.

4. R.W. Longman, "Attitude Tumbling Due to Flexibility in Satellite-Mounted Robots," *The Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 487-509.

5. W. Kohn and and K. Healey, "Trajectory Planner for an Autonomous Free-Flying Robot," *IEEE International Conference on Robotics and Automation*, San Francisco, California, April 7-10, 1986.

6. Y. Xu, H.Y. Shum, J.J. Lee, and T. Kanade, "Adaptive Control of Space Robot System with an Attitude Controlled Base," Carnegie Mellon University, The Robotics Institute, Technical Report No. CMU-RI-TR-91-14.

7. Y. Umetani and K. Yoshida, "Resolved Motion Rate Control of Space Manipulators with Generalized Jacobian Matrix," *IEEE Trans. on Robotics and Auto.*, vol. 5, no. 3, 1989, pp. 303-314.

8. E. Papadopoulos and S. Dubowsky, "On the Dynamic Singularities in the Control of Free-Floating Space Manipulators," *Dynamics and Control of Multibody/Robotic Systems with Space Applications*, edited by S.M. Joshi, L. Silverberg, T.E. Alberts, ASME, DSC-Vol. 15, 1989.

9. R.W. Longman, "The Kinetics and Workspace of a Satellite-Mounted Robot," *The Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 423-440.

10. C. Fernandes, L. Gurvits, and Z.X. Li, "Foundations of Nonholonomic Motion Planning," Technical Report No. 577, Dept. of Computer Science, New York University, August 1991.

11. H. Goldstein, *Classical Mechanics*, 2nd edition, Addison Wesley, 1980.

12. A. Ben-Israel and T.N.E. Greville, *Generalized Inverses: Theory and Applications*, New York, Krieger (1980).

13. K. Kazerounian and Z. Wang, "Global versus Local Optimization in Redundancy Resolution of Robotic Manipulators," *International Journal of Robotic Research*, Vol. 7, No. 5, 1988, pp. 3-12.

14. O.P. Agrawal, "General Approach to Dynamic Analysis of Rotorcraft," *ASCE Journal of Aerospace Engineering*, Vol. 4, No. 1, 1991, pp. 91-107.