Finding All Gravitationally Stable Orientations of Assemblies*

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Abstract

Previous work by Mattikalli *et al.*[1] considered the stability of assemblies of frictionless contacting bodies with uniform gravity. A linear programming-based technique was described that would automatically determine a single stable orientation for an assembly (if such an orientation existed). In this paper, we give an exact characterization of the *entire* set of stable orientations of any assembly under uniform gravity. Our characterization reveals that the set of stable orientations maps out a convex region on the unit-sphere of directions. The region is bounded by a sequence of vertices adjoined with great arcs. Linear programming techniques are used to automatically find this set of vertices, yielding a precise description of the range of stable orientations for any frictionless assembly.

1. Introduction

Geometric models of parts are increasingly being used to automatically plan their manufacture. Assembly of parts can also be automated using geometric models by generating high-level plans of the sequence and manner in which parts are brought together [2, 3]. To generate such plans, it is necessary to have tools that simulate the mechanics of bodies in contact. During assembly, contacting bodies are subjected to forces such as gravitational forces, contact forces and gripper/fixture forces. Since it is required that objects do not accelerate with respect to each other, a quasi-static analysis is adequate.

In Mattikalli *et al.*[1] the problem of the gravitational stability of an assembly is discussed. The assembly is modeled as consisting of frictionless contacting rigid bodies, one or more of which is assumed to be fixed in place (for example, an object held by a gripper). We will call such fixed objects *grounded*. Solutions to two problems are presented. The first problem is to determine whether an assembly in a given orientation and subject to a uniform gravity field with direction **g** is stable. An assembly is defined to be gravitationally stable if all parts of the assembly remain at rest under the influence of the gravity field. The second problem is to find an orientation of the assembly (if one exists) which is gravitationally stable. A change in orientation of an assembly refers to the rotation of all the parts in the assembly about a fixed axis, keeping **g** constant. It is simpler however to imagine the assembly as existing in some fixed orientation, and considering different gravity directions **g** which induce gravitational stability. Both problems (determining stability, and finding a stable orientation) are solvable by linear programming. However the solution method described in this work yields only a single stable orientation, even if many such orientations exist. In this respect, the method fails to provide any characterization or description of the *range* of stable orientations.

This paper presents a method to find the range of all orientations over which an assembly is gravitationally stable. We begin by first characterizing the shape of the solution space of stable orientations. This characterization shows that the set of unit gravity vectors **g** which induce stability for an assembly (in a fixed orientation) covers a convex region of the unit sphere (with the exception of one notable degenerate case). Also, if the solution space is not contained in a single hemisphere, then the solution is either an entire great circle on the sphere, or the entire surface of the sphere itself (the latter implying that the assembly is stable for *any* gravity direction). In addition to being convex, the boundary of the solution region is simply and finitely described. We will show that the shape of the solution region is the spherical analogue of a planar polygon; that is, the region's boundary is described as a sequence of vertices, with adjacent vertices connected by great arcs. Our method for describing the range of all stable orientations identifies this set of vertices on the unit sphere, using the linear programs originally described in Mattikalli *et al.*[1].

Having the entire solution space of stable orientations at hand is clearly more useful than knowing only a single stable orientation. Consider the assembly in Figure 1a, where the L-shaped part is fixed in place. This particular orientation is unstable. (For visual simplicity, our illustrations assume a fixed downwards gravity direction, and show differing orientations as an actual rotation of the assembly itself. As previously stated however, we view a change

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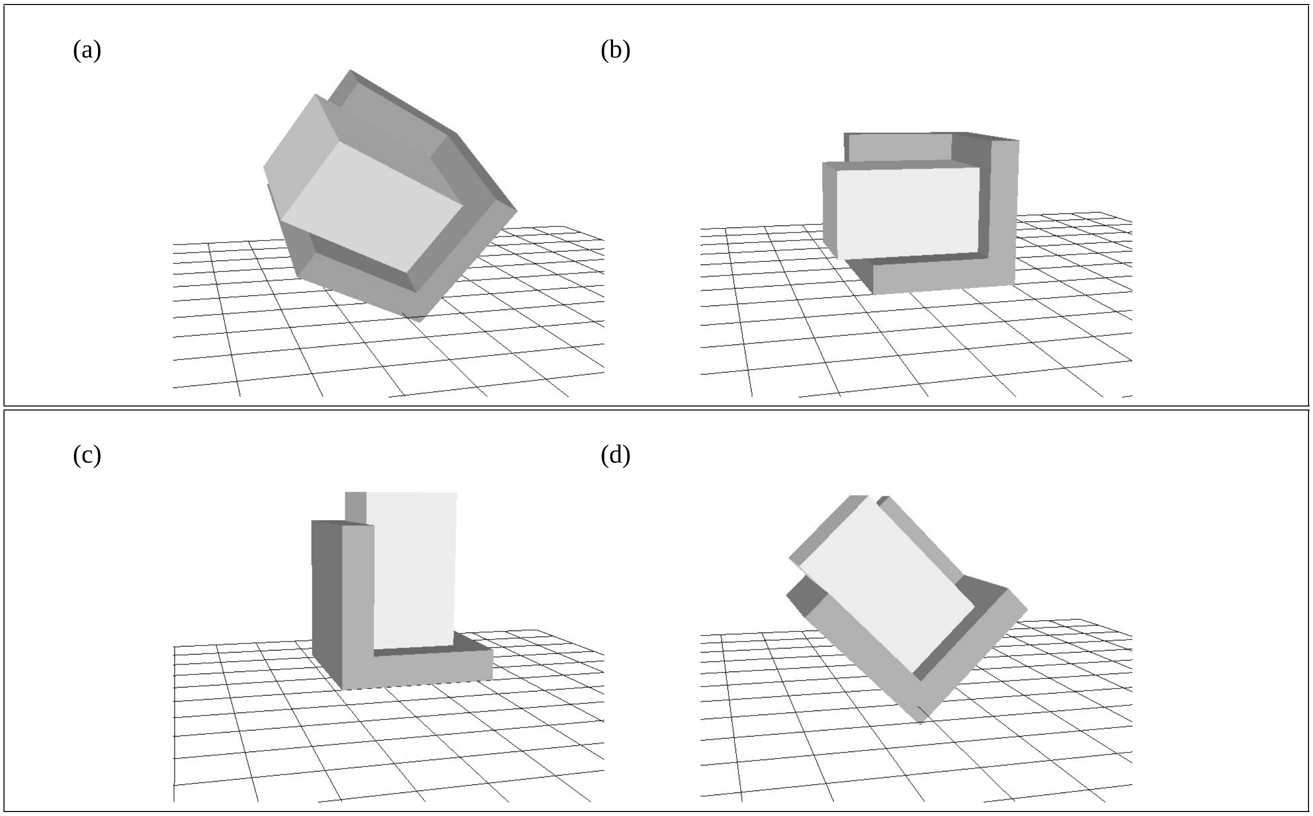

Figure 1: Finding a stable orientation. The L-shaped part is grounded. (a) Initial unstable orientation. (b) and (c) Stable orientations at bounds of stability range. (d) Stable orientation in interior of stability range.

of orientation as a variation in the gravity direction, and not the actual parts orientation.) The assembly is however stable in a range of orientations, three of which are shown in (b), (c) and (d). The orientation in (b), although stable, is at the boundary of the range of stable orientations, as is (c). A small deviation in the orientation of the assembly away from the stable region will cause the block to move. In selecting a stable orientation for this assembly, it is clear for the above reason that the orientation shown in Figure 1d is superior to those shown in Figure 1b and Figure 1c. By computing the entire range of orientations over which an assembly is stable, we will be in a position to select the most desirable one—for example one that lies at the “center” of the stable region.

In Section 2, we describe a linear program that characterizes the solution space of stable orientations for $\mathbf{g} \in \mathbf{R}^3$. We then show that the restriction of the solution space to the unit sphere implies the convexity properties described above. In Section 3 we present the details of the method to find the boundary of the region of stable orientations. We show several examples of stable regions identified for assemblies, using our method.

2. The Set of Stable Orientations

In this section, we show how the set of stable orientations $\mathbf{g}$ for an assembly may be described. At first, we will consider gravity vectors $\mathbf{g}$ of any length in $\mathbf{R}^3$ which cause the assembly to be stable. In general, this large solution set will occupy some volume in $\mathbf{R}^3$. Note that this volume will always include the solution $\mathbf{g} = (0, 0, 0)$; that is, turning gravity to zero makes any assembly stable! To discard this trivial solution, we will then intersect our larger solution volume with the unit sphere, to discover all unit-length orientation vectors $\mathbf{g}$ which result in stability. In analyzing this intersection, the convexity properties of the set of unit-length stable orientations will become apparent.

We begin by finding the set of stable orientations $\mathbf{g}$ of any length in $\mathbf{R}^3$. Figure 2 shows a set of contacting bodies placed within a gravitational field, with one of the bodies being fixed. The force of gravity acting on a part with mass M is $M\mathbf{g}$. In addition, contact forces between parts will also arise. We wish to know which gravity vectors $\mathbf{g} \in \mathbf{R}^3$ give rise to contact forces such that the net force on every part of the assembly is zero, meaning that each part is in equilibrium.

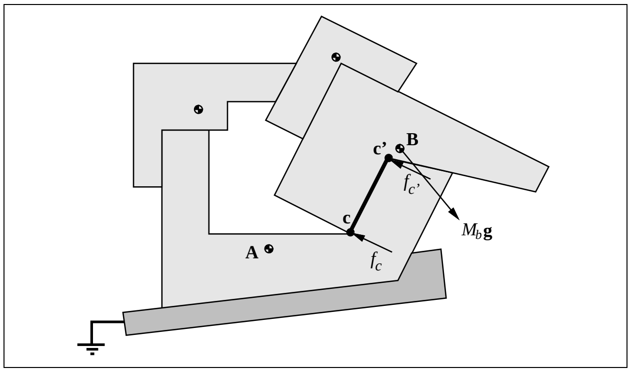

Figure 2: An assembly in a gravitational field g of unknown orientation. To find a stable g, equilibrium equations are written for each object.

2.1 Contact Forces

Since we are dealing with frictionless contacts, we know that the contact forces will act normal to the contact surfaces. We will imagine that parts contact at finitely many points, which we will index from 1 to m. (If a line segment or polygonal region of contact occurs, it is sufficient to postulate contact forces only at the vertices of the contact region.) Let $\hat{\mathbf{n}}_i$ denote the surface normal at the ith contact point between two bodies A and B, directed outwards from B. We consider a contact force $f_i\hat{\mathbf{n}}_i$, that acts on body A of the contact, and a contact force $-f_i\hat{\mathbf{n}}_i$ that acts on body B of the contact, with f_i the unknown scalar magnitude of the force. Since $\hat{\mathbf{n}}_i$ is directed from B towards A, and since contact forces must be repulsive, the magnitude f_i must be nonnegative; that is, $f_i \geq 0$.

Let the vector of contact force magnitudes f_i be denoted by $\boldsymbol{f}$. The net force $\mathbf{F}_j \in \mathbf{R}^3$ acting on the jth body of the assembly can be written as

$$\mathbf{F}_j = \sum_{i=1}^{m} s_{ji} f_i \hat{\mathbf{n}}_i + M_j\mathbf{g} \tag{1}$$

where s_{ji} is either 1, -1, or zero. If the jth body is not involved in the ith contact, then s_{ji} is zero. If the contact force exerted on the jth body from the ith contact point is $f_i\hat{\mathbf{n}}_i$, then s_{ji} is 1. Otherwise, the contact force acting on the jth body is $-f_i\hat{\mathbf{n}}_i$, and s_{ji} is -1.

The net torque $\tau_j \in \mathbf{R}^3$ acting on the jth body of the assembly is similarly written as

$$\tau_j = \sum_{i=1}^{m} s_{ji}(\mathbf{d}_i - \mathbf{c}_j) \times f_i\hat{\mathbf{n}}_i \tag{2}$$

where $\mathbf{d}_i$ is the location of the ith contact point, and $\mathbf{c}_j$ is the location of the center of mass of the jth body. The scalars s_{ji} are the same as in the previous equation. The τ_j are independent of $\mathbf{g}$ since a uniform gravity field does not exert a torque. If we define the $6n$-vectors $\mathbf{Q}$ and $\mathbf{G}$ as the collections

$$\mathbf{Q} = \begin{pmatrix} \mathbf{F}_1 \\ \tau_1 \\ \vdots \\ \mathbf{F}_n \\ \tau_n \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} M_1\mathbf{g} \\ \mathbf{0} \\ \vdots \\ M_n\mathbf{g} \\ \mathbf{0} \end{pmatrix}$$

we can write

$$\mathbf{Q} = \mathbf{A}\boldsymbol{f} + \mathbf{G}$$

where $\mathbf{A}$ is a $6n \times m$ matrix whose coefficients equations (1) and (2).

Because the assembly is frictionless, its im tion is completely determined[4]. If there ex contact forces such that the net force and tor body is zero, then such forces will arise at points, and the assembly is stable and will not a given gravity vector $\mathbf{g}$ induces stability if and exists $\boldsymbol{f}$ such that

$$\mathbf{Q} = \mathbf{A}\boldsymbol{f} + \mathbf{G} = \mathbf{0} \quad \text{and} \quad \boldsymbol{f} \geq \mathbf{0}.$$

The set of $\mathbf{g} \in \mathbf{R}^3$ for which there exists $\boldsymbol{f}$ equation (5) defines (in general) a volume V. Not the definition of $\mathbf{G}$, we have that $(0,0,0) \in V$, si $\mathbf{g}$ to zero, and all the f_i to zero clearly satisfies e Furthermore, if $\mathbf{g} \in V$, then $\alpha\mathbf{g} \in V$ for any sc since equation (5) is homogeneous. Because of solution $\mathbf{g} = (0,0,0)$ is a point in the *interior* V must be the entire space $\mathbf{R}^3$. Finally, since th described in terms of a set of homogeneous linea ties, V's boundaries (if any) are planar, passing th origin.

2.2 Characterizing Stable Orientations

Now we would like to consider the intersectio the unit sphere; that is, all gravity vectors $\mathbf{g}$ that length and induce stability. We will prove two p The first property is the following:

Property 1: If the region of stable orientatio sphere is not connected, it consists of two antipo Otherwise, the stable region covers the entire sp great circle, or is contained completely in one he

We can prove this property by considering th forms that the stable volume V might take. The possibilities.

1. The set V consists of the origin alone. In the assembly has no stable orientation. Thi satisfies property 1.

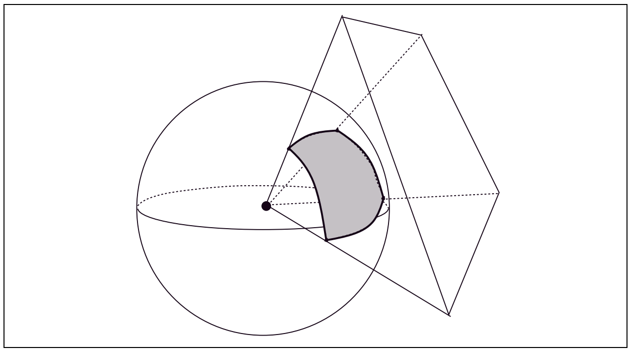

Figure 3: V **is a convex cone, whose intersection with the sphere is contained in a single hemisphere.**

2. The origin lies in the interior of V. In this case, V is equal to all of $\mathbf{R}^3$, which means that every orientation of the assembly is stable. The stable region covers the entire sphere.

3. The set V corresponds to a line passing through the origin. In this case, there are two stable orientations, represented by a pair of antipodal points on the sphere.

4. The set V corresponds to an entire plane passing through the origin. The intersection of this plane (since it contains the origin) and the sphere is a single great circle.

5. The set V is a convex cone, possibly planar (Figure 3).

The first four possibilities for V give rise to a stable region on the sphere that satisfies property 1. If V is not one of the first four cases, then V is a convex cone emanating from the origin. In this case, the intersection of V with the sphere is contained in a single hemisphere, since the cone is convex.

The second property we wish to show is the following:

Property 2: The region of stable orientations on the sphere is convex (and therefore connected), or consists of exactly two antipodal points. (We say a region on a sphere is convex if given any two points in the region, there exists a great arc between the two points that lies entirely in the region. Under this definition, a single great circle is a convex set on the sphere.)

Let us assume that the stable region on the sphere does not consist of two antipodal points. If the stable region is either empty, the entire sphere, or a great circle, then in each case the region is convex. If the stable region is none of the above, we must show that given any two points in the stable region, there is a great arc that lies between them, every point of which is itself in the stable region. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be unit vectors denoting two stable points. Since $\mathbf{g}_1$ and $\mathbf{g}_2$ are both stable orientations, they both belong to the larger set V. Since V is convex, consider the line $L \in V$ between them, given in parametric form by

$$L = \mathbf{g}_1 + t(\mathbf{g}_2 - \mathbf{g}_1) \qquad 0 \le t \le 1. \qquad (6)$$

But since for any $\mathbf{g} \in V$ and $\alpha \ge 0$ we have $\alpha\mathbf{g} \in V$, it must be that the great arc A, defined by

$$A = \frac{\mathbf{g}_1 + t(\mathbf{g}_2 - \mathbf{g}_1)}{\|\mathbf{g}_1 + t(\mathbf{g}_2 - \mathbf{g}_1)\|_2} \qquad 0 \le t \le 1 \qquad (7)$$

is also in V, and hence stable. Note that by property 1, the stable region is contained in a single hemisphere, so $\mathbf{g}_1$ and $\mathbf{g}_2$ cannot be antipodal. This means $\mathbf{g}_1 + t(\mathbf{g}_2 - \mathbf{g}_1)$ is always nonzero, so that A is well-defined. Since A lies on the unit sphere, and also in V, every point in A is by definition in the stable region. This proves property 2.

As a further note, since V's boundary are planes passing through the origin, the intersection of those planes with the unit sphere are great arcs, and form the boundary of the stable region on the sphere (which we might term a "spherygon.") In the next section, we will show how the vertices of this "spherygon" are found.

3. Finding the Vertices of the Stable Region

We now show how the vertices of the stable region "spherygon" may be found. Essentially, we are looking for the extremal points $\mathbf{g} \in V$, restricted by $\|\mathbf{g}\|_2 = 1$. We can greatly simplify matters if instead of intersecting V with the curved surface of a sphere, we intersect V with the flat surfaces of a unit cube. That is, we consider solutions to equation (5) with the restriction $\|\mathbf{g}\|_\infty = 1$. (For a vector $\mathbf{v}$, $\|\mathbf{v}\|_\infty = \max_i |v_i|$.) In this case, the intersection of V with any single face of the unit cube is a polygon (or degenerate polygon), rather than the more complex "spherygon." The solutions $\mathbf{g}$ which are the vertices of this polygon, will, when normalized to unit length, be the vertices of the "spherygon." The advantage of this transformation is that it makes the entire problem a problem in linear inequalities, enabling us to make use of linear programming techniques.

Thus, let us consider instead solutions to equation (5) such that $\|\mathbf{g}\|_\infty = 1$. Furthermore, in what follows, we will consider only a single face of the unit cube $\|\mathbf{g}\|_\infty = 1$. Without loss of generality, in the remainder of this paper, we will further restrict $\mathbf{g}$ by setting $g_x = 1$ and imposing the constraint that $-1 \le g_y, g_z \le 1$. Any statements made hereafter about $\mathbf{g}$ assume that they imply in turn to the other five possibilities $g_x = -1$ and $-1 \le g_y, g_z \le 1$, $g_y = 1$ and $-1 \le g_x, g_z \le 1$ etc.

Our problem is now to find the vertices of the region in the g_y, g_z plane that satisfy

$$\mathbf{Q} = \mathbf{A}\boldsymbol{f} + \mathbf{G} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{f} \ge \mathbf{0} \qquad (8)$$

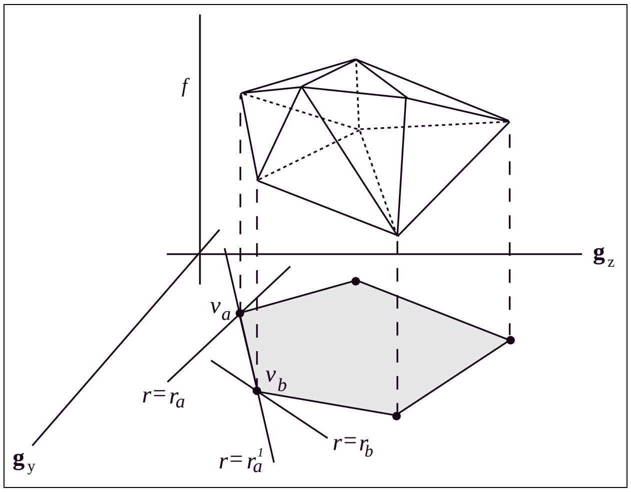

Figure 4: The projection of the feasible region onto the $g_y g_z$ plane. This projection defines the range of stable orientations.

given the constraints $g_x = 1$ and $-1 \le g_y, g_z \le 1$. If we consider the set of feasible solutions $F = (g_y, g_z, \boldsymbol{f})$ to this equation, we are trying to find the vertices of the projection of F onto the (g_y, g_z) plane (Figure 4).

Our technique to find the vertices of this projection is as follows: we find an initial vertex of the projection, and then use the technique of *parametric programming* to traverse the perimeter of the projection. Finding an initial vertex of the projection is simple: we first find a solution $(g_y, g_z, \boldsymbol{f})$ which minimizes g_z (always subject to $-1 \le g_y, g_z \le 1$). Note that we are not interested in the value of $\boldsymbol{f}$ when g_z is minimized; we are only interested in the coordinates (g_y, g_z) at which the minimum occurs. If this first minimization does not uniquely determine both g_y and g_z, we temporarily hold g_z fixed, and find a solution $(g_y, g_z, \boldsymbol{f})$ which minimizes g_y. In this way, we find some point in F whose projection is a vertex of the projection of F onto the g_y, g_z plane. We will call this initial vertex v_a. (Note that in Figure 4, the vertex v_a is determined with only a single minimization on g_z.)

Having found an initial vertex of the projection of F, we now need to traverse the perimeter. We found our initial vertex (g_y, g_z) of the projection by minimizing a function of the form

$$C_1 g_y + C_2 g_z. \tag{9}$$

Let us assume that simply minimizing g_z was sufficient to find the initial vertex v_a. In that case, we have performed a minimization with $C_1 = 0$ and $C_2 = 1$ to find v_a. Suppose that we begin to increase C_1. While C_1 is close to zero, the vertex of the projection we obtain by minimizing equation (9) will still be v_a. However, for some value of C_1, the vertex minimizing equation (9) will abruptly shift to vertex v_b (assuming that the positive g_y axis is directed into the page). Geometrically, the line r_a corresponds to choosing $C_1 = 0$ and $C_2 = 1$. As we increase C_1, the line r_a swings towards the line r_a^1. At the point that the line r_a hits r_a^1, we find another vertex v_b (in addition to v_a which minimizes equation (9)). We can determine this limit using *sensitivity analysis*, a technique of linear programming. As we continue swinging the line past r_a^1, vertex v_a ceases to become a minimizer, and the values of g_y and g_z at v_b are the sole minimizer of equation (9). By selecting v_b as the new optimal point, further sensitivity analysis can be done to find the next vertex past v_b, and the process can be continued. At some point of course, we will reach a point where increasing C_1 never reaches a vertex, and we will need to switch to changing the value of C_2. Eventually, we will find ourselves back at the original vertex v_a, and the traversal is stopped. Note that procedure this is quite similar to the manner in which the simplex algorithm for linear programming itself works.

Having found the vertices of the stable region on all faces of the unit cube, it is trivial to transform those vertices to the unit sphere and map out the stable region on the sphere. In Figures 5 and 6 we present some sample assemblies and their stable regions shown shaded on the surface of a sphere. In each figure, on the left is shown an assembly along with a coordinate system. In both the figures the larger of the two objects is grounded. On the right, a wireframe of a sphere is shown, with the stable region shown as shaded patches. The coordinate system shown with the sphere is aligned with that shown with the assembly. Figure 5 consists of a cube sitting within a concavity in a grounded block. If the block is oriented such that the 3 edges on the block that possess an internal angle of $3\pi/2$ are along the 3 positive coordinate directions, then the stable region is defined by the 3 negative coordinate directions. This region can be see as the shaded region on the far side of the sphere.

Figure 6(a) shows a block that is placed within a grounded L-shaped block. The stable region corresponds to a line as shown on the sphere in Figure 6(b).

Figure 7 shows an example where the stable region is less than an entire facet. The large parts with 3 hollows is the grounded part. It is shown separately for clarity in Figure 7(a). Figure 7(b) shows the assembly wherein 3 cubes are placed within the 3 hollow cubical spaces, such that each cube make contact with the grounded part over 3 faces. The stable region is shown in Figure 7(c) in the same orientation as the assembly in Figure 7(b).

4. Conclusion

In modeling assemblies to plan for product manufacture, it becomes necessary to be able to predict their mechanics under external forces such as gravitational forces, fixturing forces and machining forces. Most often it is desired that

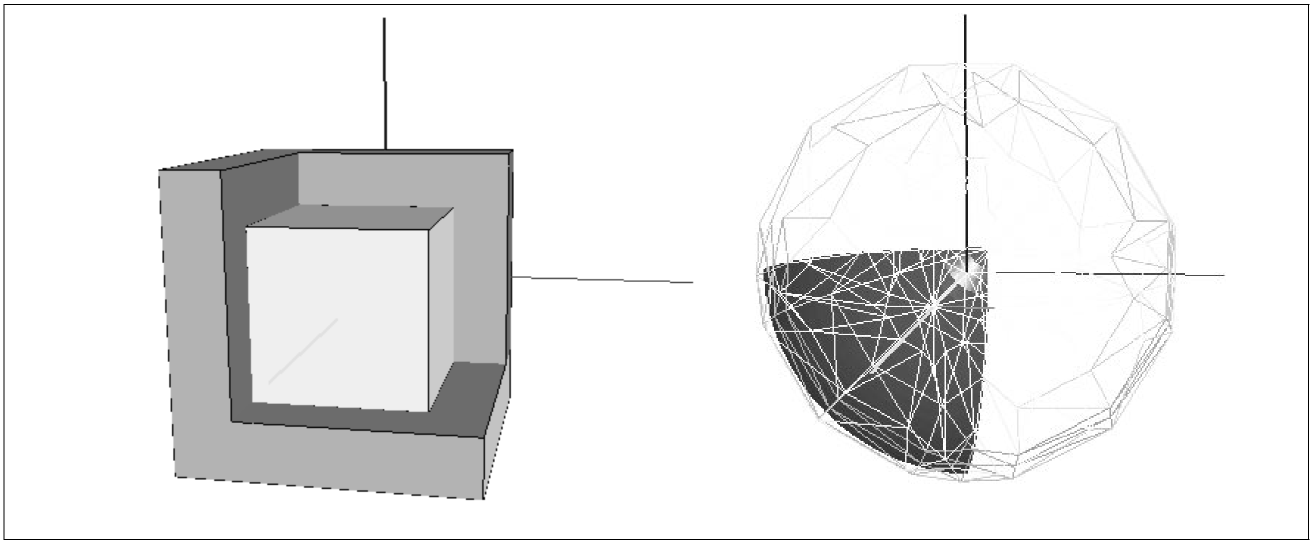

Figure 5: (a) An assembly with the large block grounded. (b) The shaded region on the far side of the sphere (a complete octant of the sphere) indicates all stable orientations.

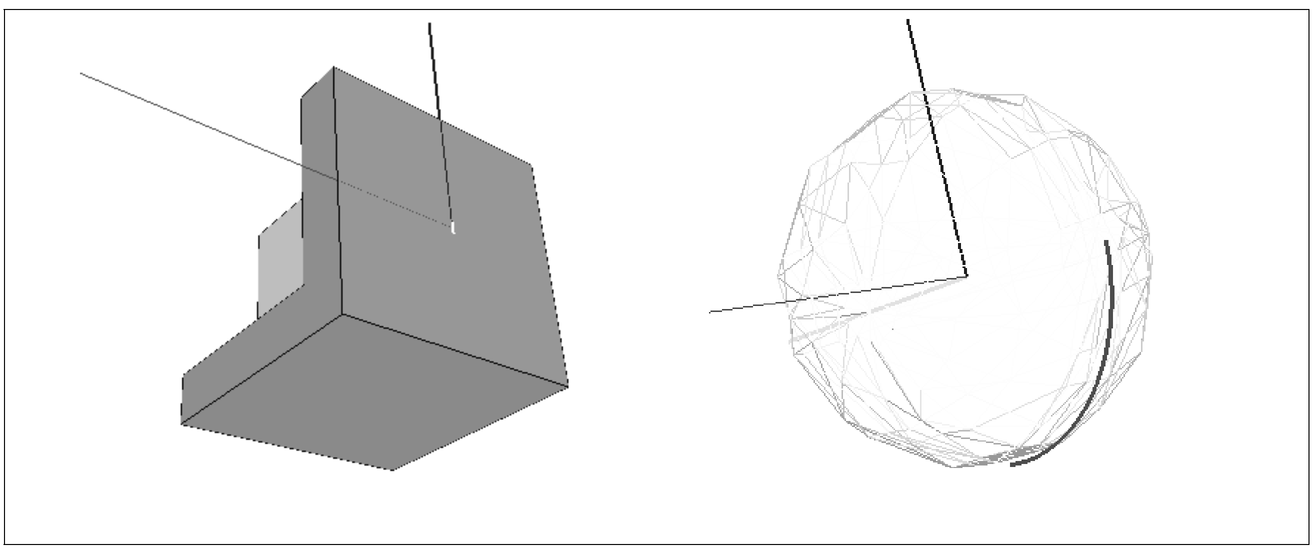

Figure 6: (a) An assembly with the L-shaped block grounded. (b) The dark line on the sphere shows all stable values for g.

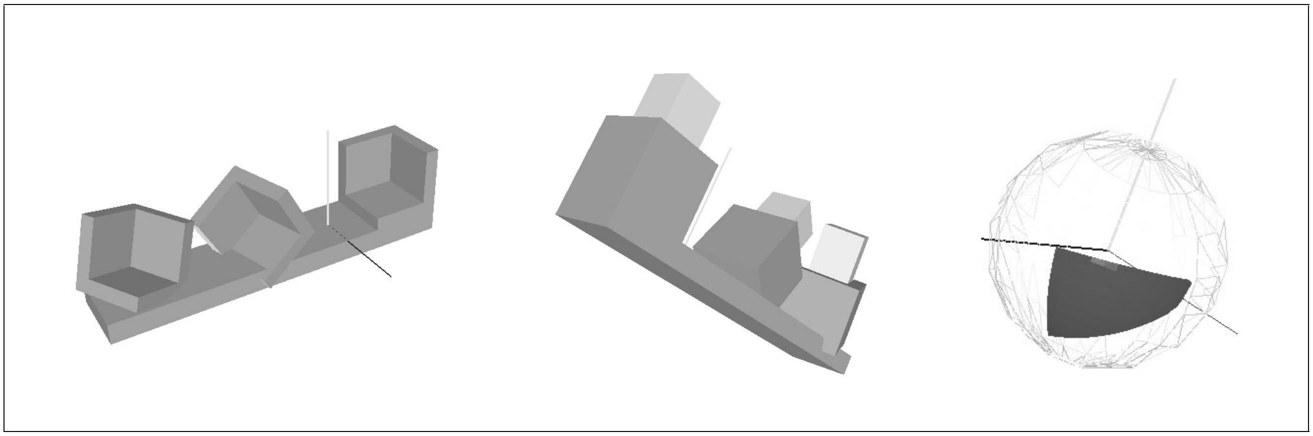

Figure 7: (a) The grounded part of an assembly. A cube is placed in each of the cubicals. (b) The assembly with the 3 cubical parts. (b) The shaded region on the sphere shows all stable values for g.

the parts remain motionless. Assuming frictionless rigid bodies and a single fixed body, in this paper we present a method of finding *all* the stable orientations in which an assembly can be placed within a uniform gravity field. The method extends earlier work which found a single stable orientation. To find all stable orientations, the proposed method solves for the projection of the entire set of feasible solutions of a high-dimensional linear program onto a two-dimensional plane. Stable orientations are mapped onto regions on the surface of a sphere. For a given assembly, the mapping of stable regions onto the surface of a sphere is graphically displayed. This mapping can be used during assembly process planning. Any subassembly that is being manipulated, or placed on a table, or within a fixture, can now be placed in an orientation that keeps the subassembly stable. The dimension of the linear program is given by the number of contact points, so that an assembly with 100 contact points is solvable in well under a second of CPU time.

Acknowledgements

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