A Computational Representation for Rigid and Articulated Assembly

Yanxi Liu

CMU-RI-TR-97-66

The Robotics Institute
Carnegie Mellon University
Pittsburgh, PA 15213

A shorter version of this paper appears in the proceedings of 1997 IEEE International Symposium on Assembly and Task Planning (ISATP97).

1 Introduction

With the increasing level of automation in assembly planning and assembly execution by machines and robots, it becomes more obvious that there is a gap between the output of a mechanical designer and the input of an assembly planner. The question is: How to describe a designed assembly to an assembly planning system? The input to almost all the current reported automatic assembly planning systems [17, 18, 11, 5] is one-static-state of the final assembly configuration regardless the assembly is meant to be rigid or articulated. The inability to represent the assembly design completely, accurately and computationally has hindered the power of an assembly planner in dealing with articulated assemblies as simple as taking something out of a drawer. In this paper we identify a computational representation (specification) of an assembly that is composed of rigid solids[1]. The basic idea of this representation is simply to use each *oriented* surface on a solid as its primitive feature and each primitive feature is attached with its symmetry group. The relative motions (degree of freedom) under various contacts between a part or a subassembly and the rest of the assembly can be efficiently determined by combining these basic symmetry groups in a certain way. Different from the study of solids in local contact [6, 13] etc., our aim is to have a precise and complete description of the intended, possibly articulated, final assembly configuration where each part usually has multiple contacts with the rest of the assembly; and our approach is algebraic in nature. Different from our previous work [9] where the surfaces of a solid are treated as set points without taking orientations into consideration, in this work oriented surfaces are used as the basic building blocks. Also different from [14, 15, 16] in that a group theoretical formalism is embedded in a concise representation of an assembly not involving extensive algebraic equation manipulation.

In Section 2 we establish the basic vocabulary — oriented surfaces and their symmetry groups — for describing assembly, and the relationship of a pair of oriented surfaces. Then in Section 3 we introduce compound feature as a result of considering multiple contacting features between solids. The results on computing the symmetry group of a set of oriented surfaces are derived. In Section 5 we summarize current results and discuss future work. All proofs are in the Appendix section (Section 6).

[1] A three dimensional continuum for which the distance between any pair of its points remains unchanged under any physically possible motion

2 Basics: Oriented Primitive Feature and Its Symmetry Group

Since contacts among solids happen via the contacts of the surfaces of the solids, the representation and characterization of each surface constitutes the foundation of any formalization for solid contacts.

2.1 Set Surface

In [9] a group theoretic framework was proposed where surfaces of a solid treated as subsets in Euclidean space. Under such a formalization, general yet concise expressions for computing relative positions between solids from information on surface contacts are induced. A set-feature of a solid is defined as:

Definition 2.1.1 *A* **primitive feature** F *of a solid* M *is a connected, irreducible and non-trivial algebraic surface that partially or completely coincides with one or more finite bounded faces of* M.

A proper symmetry for such a set-feature is defined as:

Definition 2.1.2 *A proper isometry*[2] g *is a* **proper symmetry of a set** $S \subset \Re^3$ *if and only if* $g(S) = S$.

All the symmetries of a set-feature form a *group* mathematically, and is thus called *the symmetry group of a feature*.

It is true that in general whether a surface has orientations or not, or which orientation it has, does not make a difference in regards to the symmetries of the feature[3]. A spherical surface, treated as a set or with orientation vectors pointing inward, has the same symmetries as the spherical surface with orientation vectors pointing outward. However, in real world problems it is rare that only one surface is considered in isolation. In an assembly, it is often the case that multiple surfaces of one solid are in contact with multiple surfaces of other solids. This is a situation where a surface is treated as a set can run into problems.

For example, Figure 1 shows two adjacent planar surfaces S_1, S_2 of a block. If the two features are treated as sets the symmetries of the two planes include a 90^0 rotation about the line of the intersection of the two planes which is not a symmetry in reality. If one takes into consideration

[2] A proper isometry is a distance and handedness preserving mapping.

[3] The only exception is the planar surface: when it is treated as a set there are flipping symmetries which do not exist for oriented planes. In practice, however, this can be easily handled by treating planes as the only special case.

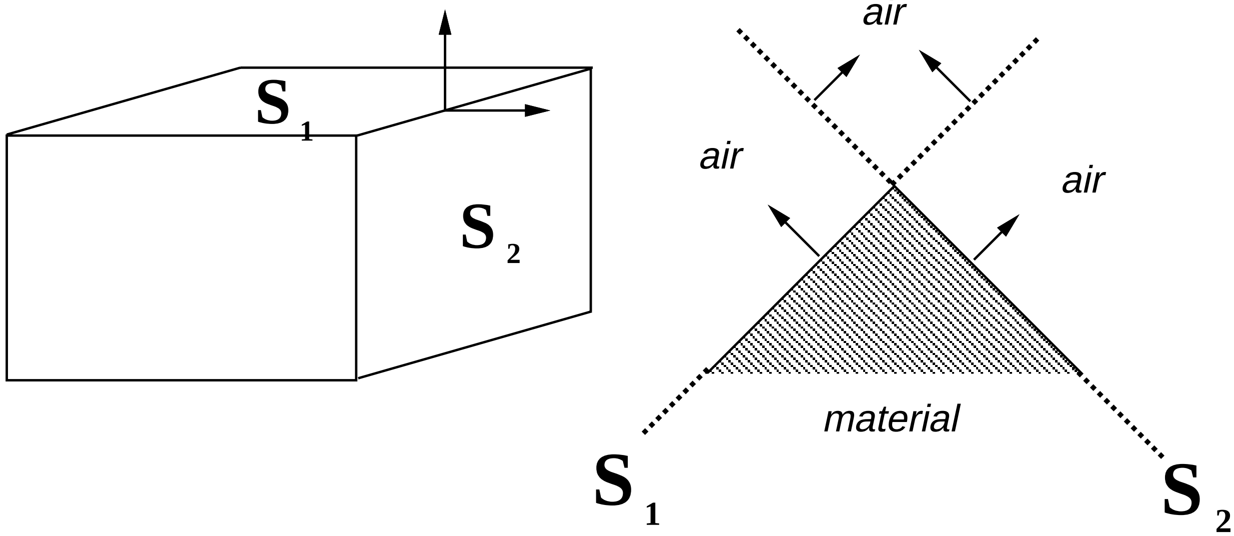

Figure 1: Two adjacent planes, S_1, S_2, on a cube

the fact that one side of the plane is the material of the solid and the other is the air, the symmetries only contain those 180^0 rotations that preserve both the bounding surfaces and their orientations.

Another example of such non-real symmetries is illustrated in Figure 2. If the two cylindrical surfaces S_1, S_2 are treated as sets then one cannot distinguish the two cases (a) and (b). In case (b) the cylindrical hole S_1 and the cylinder S_2, though they have the same radius, are not interchangeable if one takes their orientations into consideration.

Because having the same symmetry group is a necessary condition for a pair of solids to have a surface contact, obtain the accurate symmetry group of a set of contacting surfaces becomes crucial in applications such as assembly planning where the planner needs to decide which assembly parts fit with each other [10, 7].

The aforementioned problems call for a more precise characterization of surface features of a solid, i.e. taking the orientations of a surface into consideration. This addition to a set-feature will require that the symmetries of the feature keep both the points on the surface and the orientations of the surface, respectively, setwise invariant.

2.2 Oriented Surface and Its symmetry Group

The surfaces which we have treated mathematically as subsets of $\Re^3$ have no intrinsic *inside and outside*. To remedy this we introduce the concept of *oriented features* by defining a set of outward-pointing normal vectors for each surface point of a solid. The polynomial used to express an algebraic surface implicitly defines such normal vectors. Let $\mathcal{S}^2$ be the unit sphere at the origin

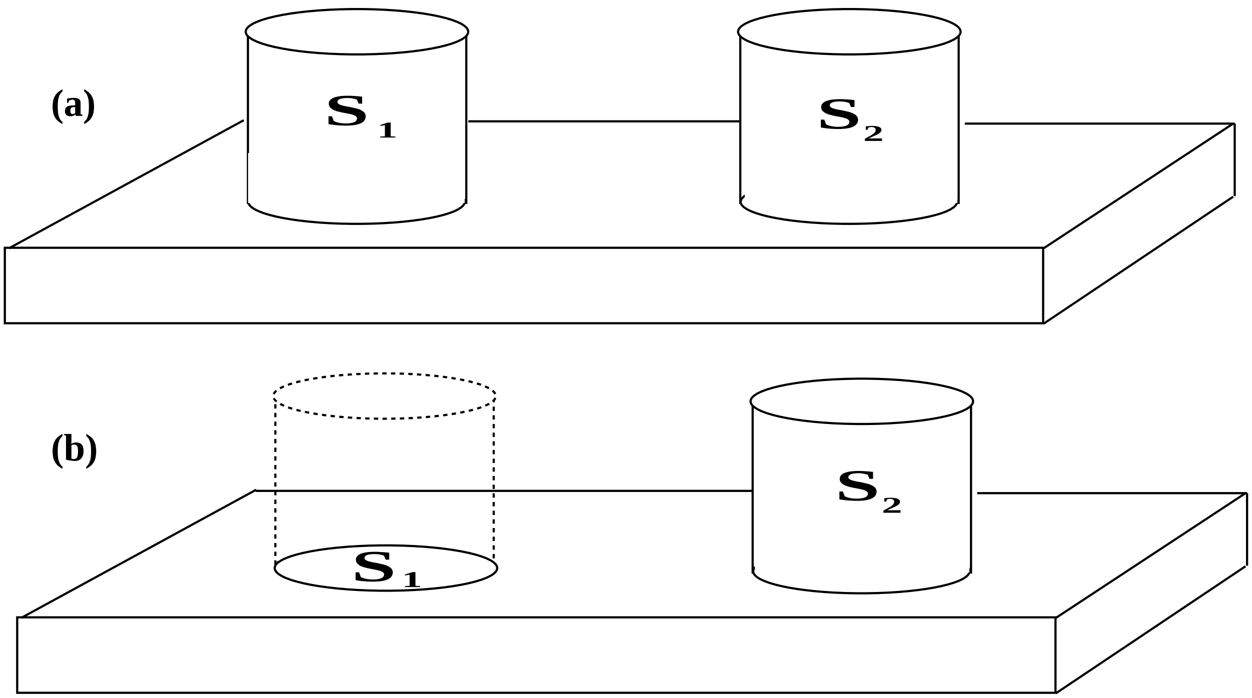

Figure 2: S_1 and S_2 are interchangable if they are treated as sets.

embedded in $\Re^3$, each point of $\mathcal{S}^2$ corresponds to a unit vector in $\Re^3$.

Definition 2.2.1 *A solid M is a connected, rigid, three dimensional subset of Euclidean space $\Re^3$.*

Definition 2.2.2 *An* **oriented primitive feature** $F = (S, \rho)$ *of a solid M is an oriented surface where*

1) $S \subset \Re^3$ is a connected, irreducible[4] *and continuous algebraic surface which partially or completely coincides with one or more finite oriented faces of M;*

2) $\rho \subset S \times \mathcal{S}^2$ is a continuous relation. For each $s \in S$ if s is a non-singular point of surface S (p.78 [3]) then $v \in \mathcal{S}^2$ is one of the two opposing normals of the tangent plane at point s such that $(s, v) \in \rho$; if s is a singular point of S (e.g. at the apex of a cone) then, for all v, where $v \in \mathcal{S}^2$ is the limit of the orientations of its neighborhood, $(s, v) \in \rho$.

3) For all $s \in M, (s, v) \in \rho, v$ points away from M.

Intuitively speaking, a feature is composed of both “skin", S, and “hair", the set of normal vectors which correspond to the points on $\mathcal{S}^2$. Each element of relation ρ is a correspondence

[4] Here *irreducible* implies that a primitive feature cannot be composed of any other *more basic* surfaces.

between a point on S and a vector on $\mathcal{S}^2$. Note, there may be more than one 'normal vector' at one point of a surface, e.g. at the apex of a conic shaped surface.

Let $\mathcal{E}^+$ be the proper Euclidean group which contains all the rotations and translations in $\Re^3$, and $\mathbf{T}^3$ be the maximum translation subgroup of $\mathcal{E}^+$. We now define how an isometry acts on the relation ρ defined in Definition 2.2.2:

Definition 2.2.3 *Any isometry $g = tr$ of $\mathcal{E}^+, t \in \mathbf{T}^3, r \in SO(3)$ acts on ρ in such a way that $(s, v) \in \rho \Leftrightarrow (gs, rv) \in g * \rho$.*

Now we define the symmetries for an oriented surface:

Definition 2.2.4 *An isometry $g \in \mathcal{E}^+$ is a* **proper symmetry of a feature** $F = (S, \rho)$ *if and only if $g(S) = S$* **and** *$g * \rho = \rho$.*

Note, the difference between the symmetries of a set (Definition 2.1.2) and this definition. There is an extra demand on a symmetry for an oriented feature — it has to preserve the orientations of the feature as well. Since orientations are points on $\mathcal{S}^2$, symmetries of an oriented feature have to keep **two** sets of points in $\Re^3$ setwise invariant. One can prove that the symmetries for an oriented surface form a group as well:

Proposition 2.2.5 *The symmetries of an oriented feature $F = (S, \rho)$ form a subgroup of $\mathcal{E}^+$, called the* **symmetry group of feature** F.

Proof :

Let G denote the set of the symmetries of F. Since it has been shown in Proposition 6.0.18 that it is true for set S, here we only state about ρ.

Obviously, $1 * \rho = \rho$, so $1 \in G$. If $g \in G$ then $(g * \rho) = \rho$ (By the definition of symmetries). Multiplying by g^{-1} we have $g^{-1}(g * \rho) = g^{-1} * \rho$. Using Lemma 6.0.17 we have $g^{-1} * \rho = \rho$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2) * \rho = g_1 * (g_2 * \rho) = g_1 * \rho = \rho$ therefore $g_1 g_2 \in G$. Hence G is a subgroup of $\mathcal{E}^+$. $\square$

3 Multiple Contacts: Compound Features and their Symmetry Groups

An assembly is a manifestation of surface interactions of its subparts, albeit the physical property of each individual part (rigid or deformable) or the nature of the contact (static or articulated). Thus the representation of an assembly is reduced to how to specify a set of contact constraints which dictate the configuration of a set of solids.

From [7, 9] we have developed an expression for the relative motions of two solids B_1 and B_2 in contact via surfaces F_1 and F_2 respectively:

$$l_1^{-1} l_2 \in f_1 G_1 G_2 f_2^{-1}, \tag{1}$$

where $l_1^{-1} l_2$ is the relative position of solid 2 w.r.t. solid 1, G_1, G_2 are symmetry groups of F_1 and F_2 respectively, l_1, l_2 specify the locations of solids B_1, B_2 in the world coordinate system and f_1 and f_2 specify the locations of F_1, F_2 in their respective body coordinates (Figure 3).

A more specific form for the relative positions of two solids under n surface contacts:

$$l_1^{-1} l_2 \in f_1 G f_2^{-1} \tag{2}$$

has shown clearly that the possible motions of a solid or a subassembly S in an assembly are described precisely by the symmetry group G of the multiple contacting oriented surfaces of S. If G is an identity group, i.e. $l_1^{-1} l_2 = f_1 f_2^{-1}$ give a fixed position for S. If G is a finite rotation group, then $f_1 G f_2^{-1}$ contains a finite number of positions reflecting the existence of finite symmetry of the contacting surfaces. If G is a continuous group then there exists relative continuous motions between S and the rest of the assembly.

Let us first give a denotation for such a set of contacting surfaces, and then determine what the symmetry group of this collection of surfaces should be.

Definition 3.0.1 *A* **compound feature** $F = (S, \rho)$ *of primitive features* $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$*, is defined to be*

- $S = S_1 \cup ... \cup S_n$
- $\rho = \rho_1 \cup ... \cup \rho_n$

The advantage of using a relation ρ to denote the orientations of a feature (Definition 2.2.2) becomes more obvious for compound features. When two primitive features are combined, there often are multiple normal directions at the points where the surfaces meet. For example , at the bottom of a cylinder where a cylindrical surface and a planar surface intersect, there are two normals for each point at the intersection (Figure 4).

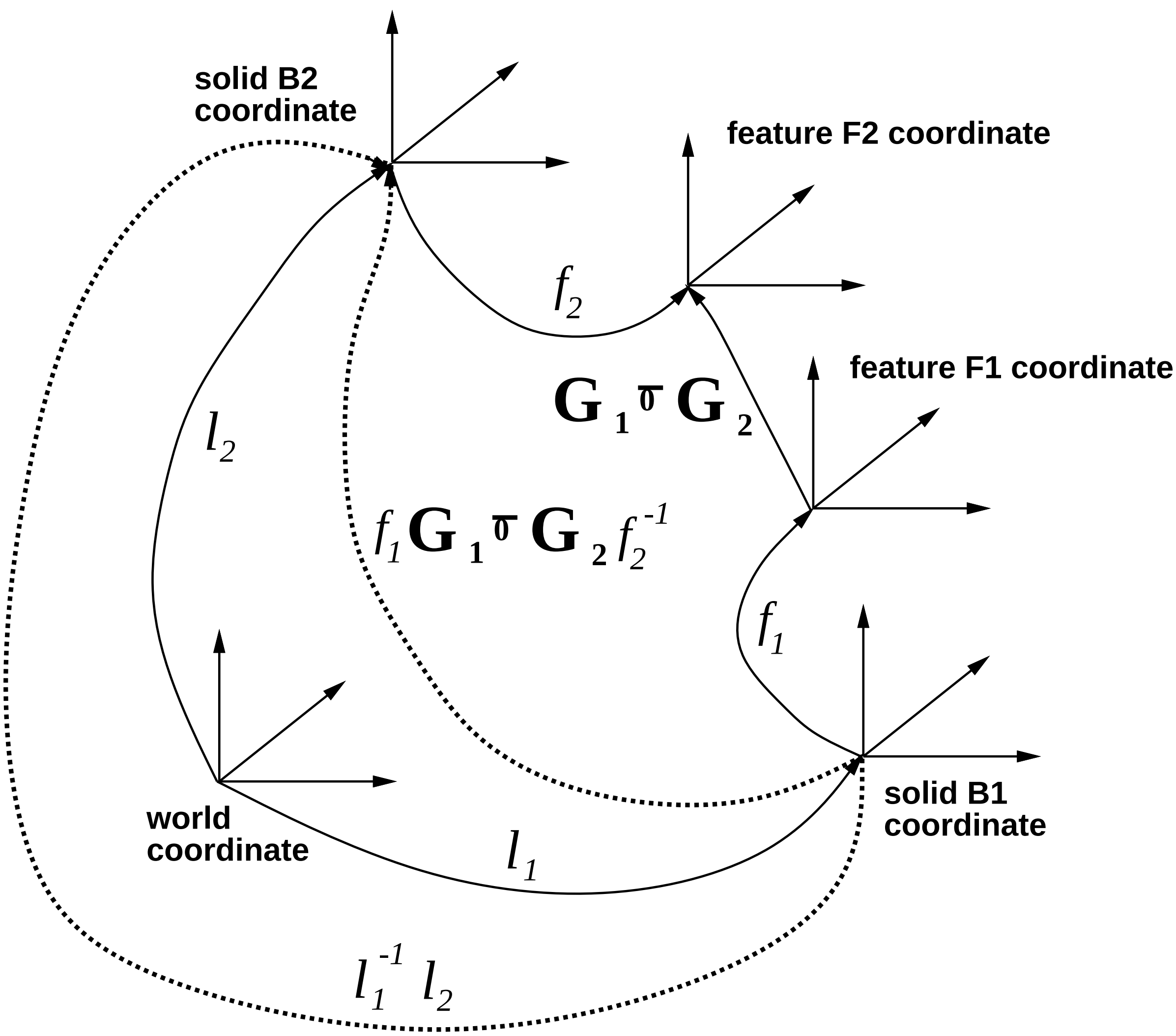

Figure 3: The relative locations of two solids B_1, B_2, in contact through their features F_1, F_2, are expressed in terms of their body coordinates and the symmetry groups of the contacting features.

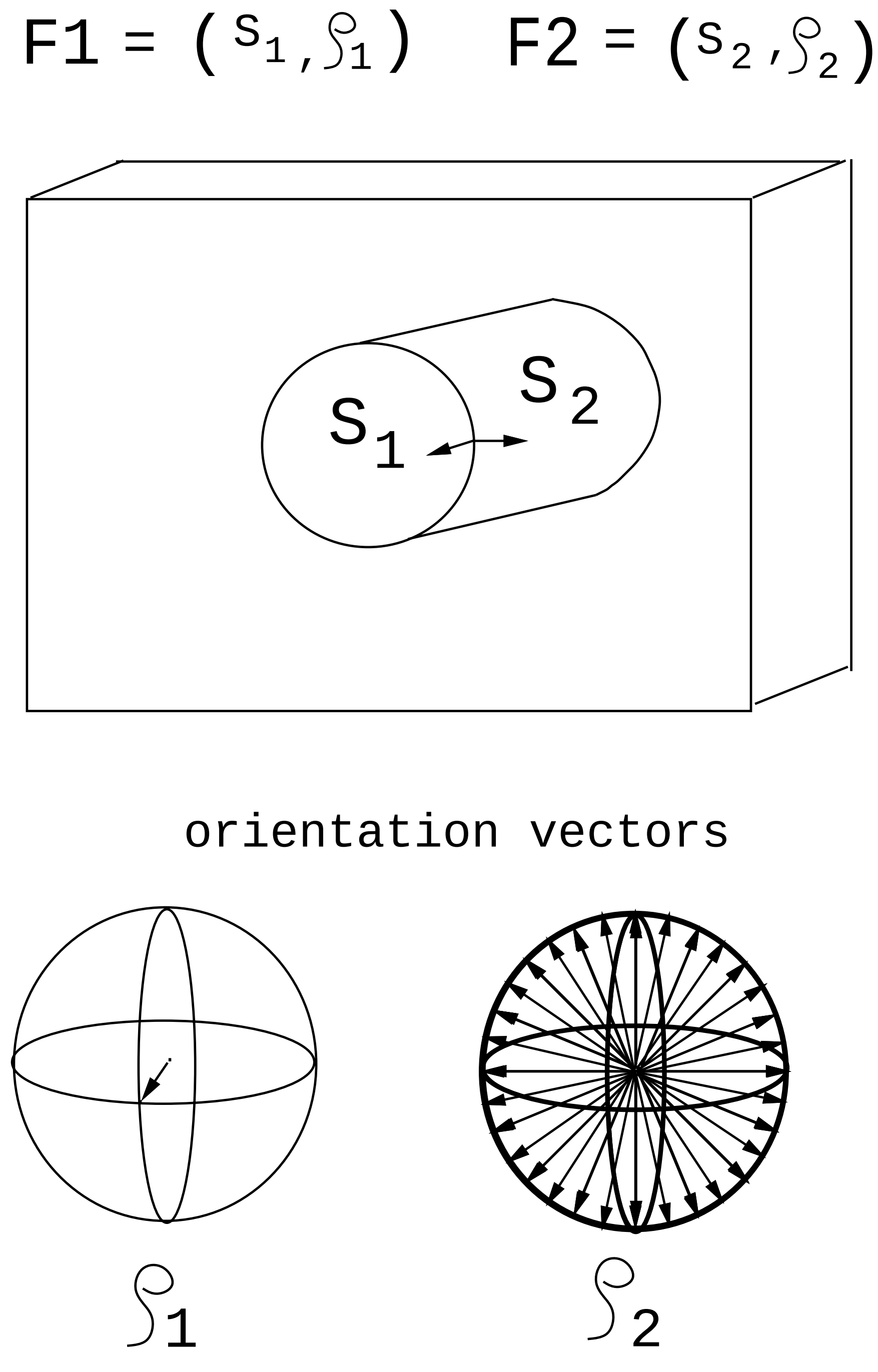

Figure 4: A pair of **distinct** features F_1, F_2

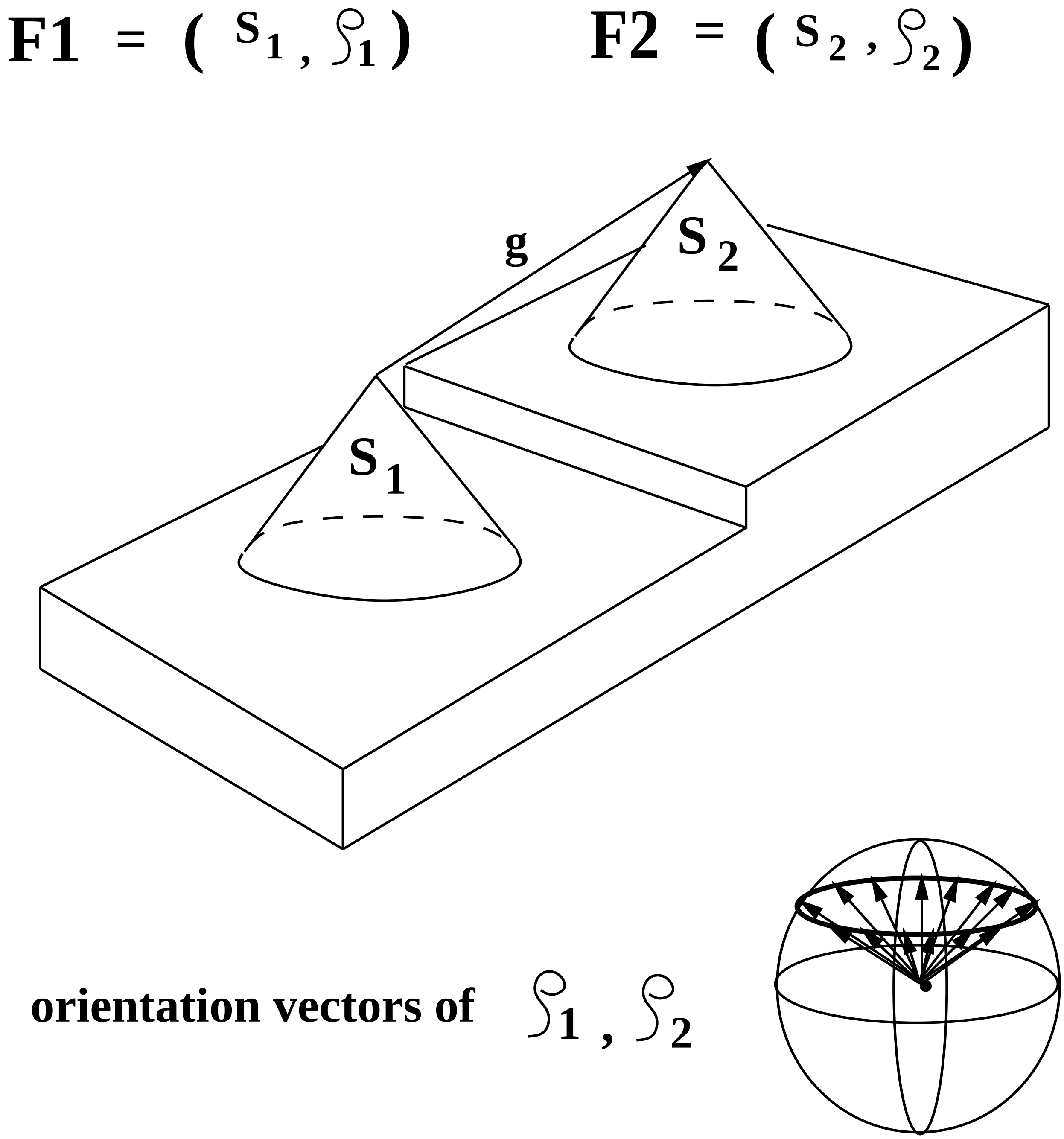

Figure 5: Two conic features F_1, F_2 which are **1-congruent** to each other

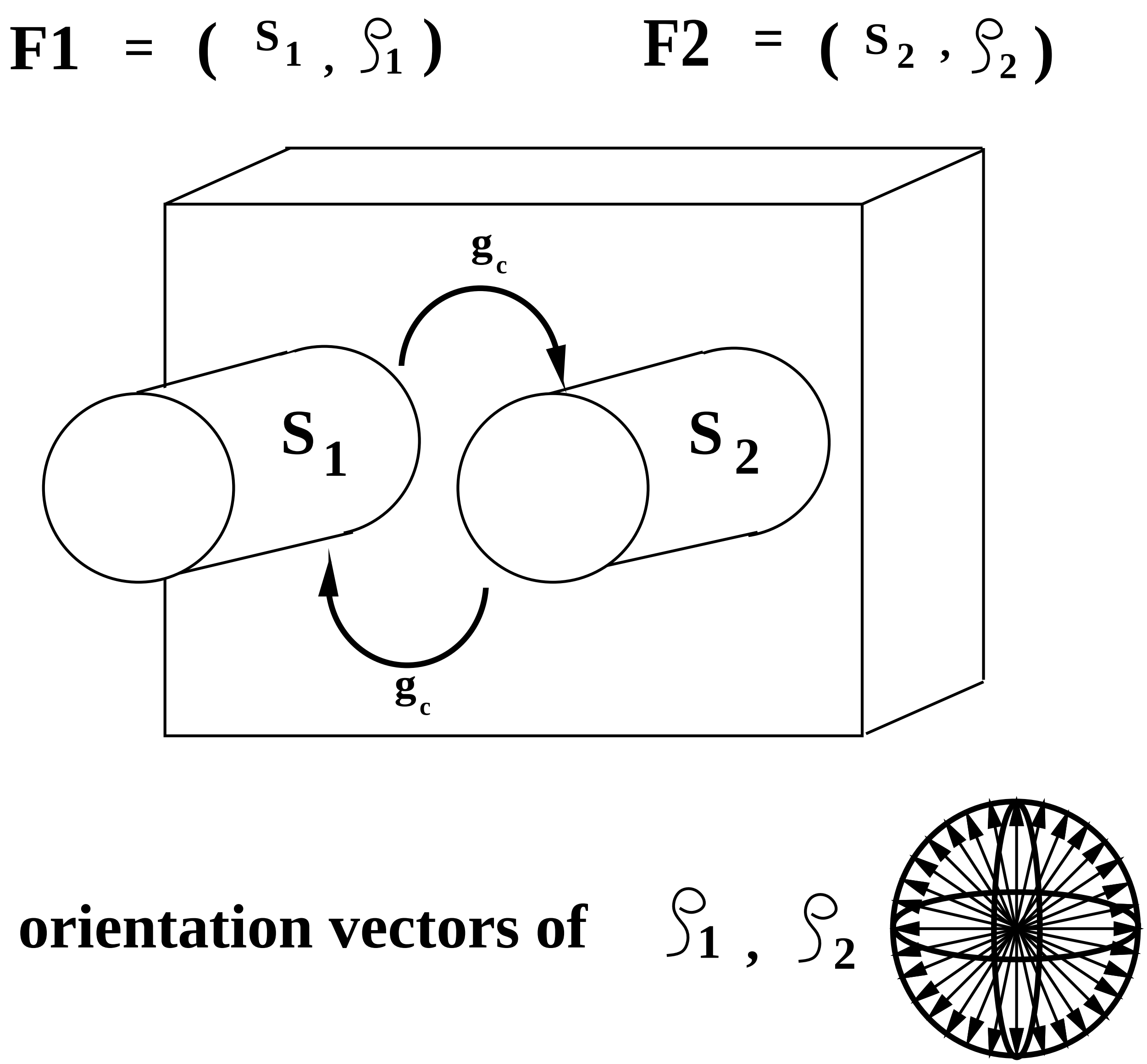

Figure 6: Two cylindrical features F_1, F_2 which are **2-congruent** to each other

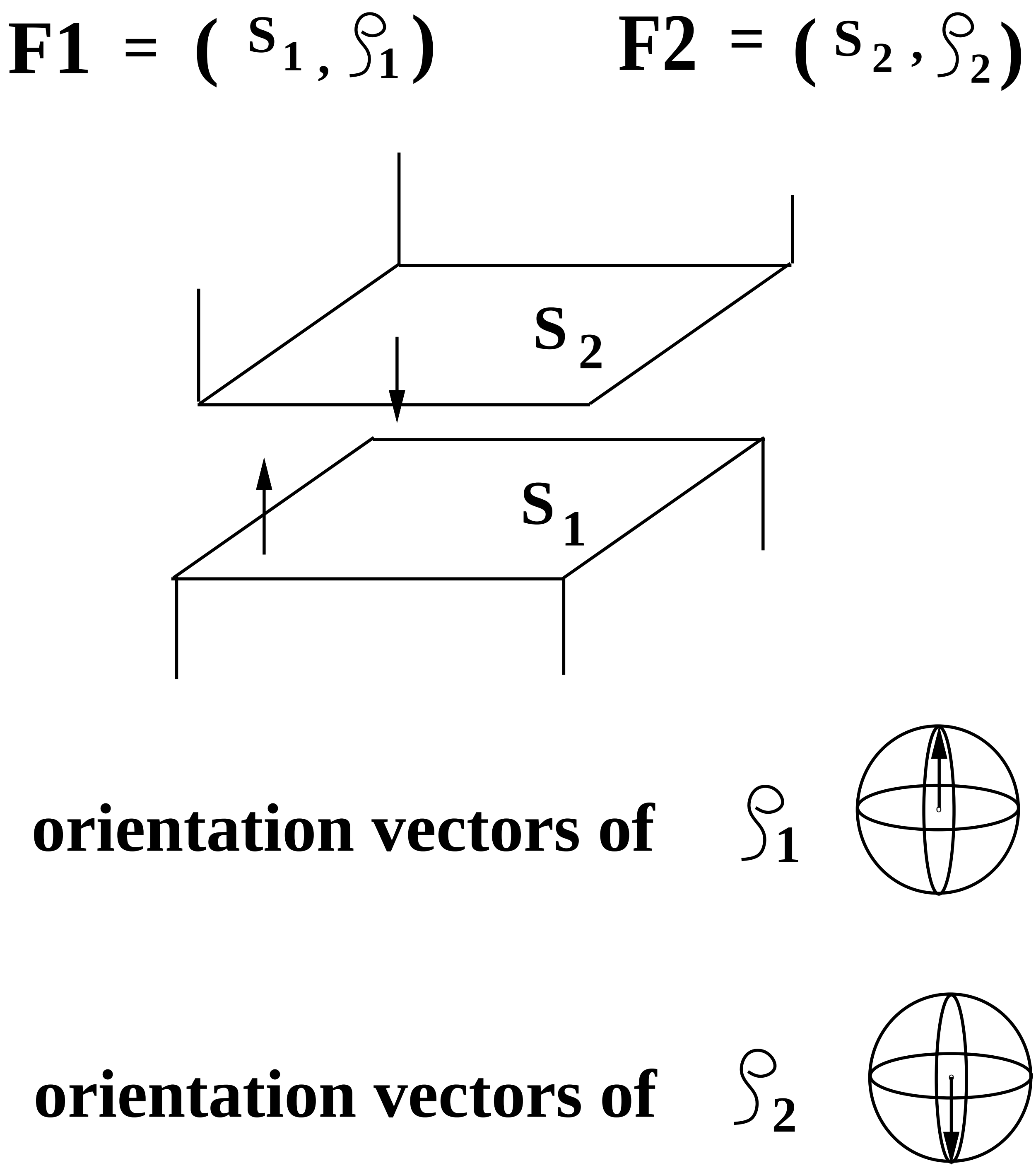

Figure 7: Two complementary features F_1, F_2

3.1 Pairwise Relationship of Oriented Features

In order to determine the symmetry group of a compound feature systematically, we start with the simplest compound feature — a compound feature composed of only one pair of primitive features. See Figure 4, 5 and Figure 6 for examples of these simple compound features (Note that only a finite face on each primitive feature is drawn).

Given a pair of primitive features, what kind of relationship holds between the two features and what is the effect of such a relationship in terms of determining their symmetry group? The following definition gives a characterization of four relationships between a pair of primitive features:

Definition 3.1.2 *Two oriented primitive features* $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ *are said to be*

- **Distinct:** if for any open subsets $S'_1 \subset S_1, S'_2 \subset S_2$, no $g = tr \in \mathcal{E}^+$ exists such that $g(S'_1) \subset S_2$ or $g(S'_2) \subset S_1$. See Figure 4 for an example of a pair of distinct features F_1, F_2.

- **1-congruent**: if there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = \rho_2$, but for all such $g, g(S_2) \neq S_1$. For an example see Figure 5. Another example is two parallel planar surfaces with normal vectors pointing in the same direction.

- **2-congruent**: if there exists $g_c \in \mathcal{E}^+$ such that $g_c(S_1) = S_2, g_c(S_2) = S_1, g_c * \rho_1 = \rho_2$ and $g_c * \rho_2 = \rho_1$. For an example, consider two parallel cylindrical surfaces having the same radius and normal vectors pointing away from their center lines, as in Figure 6. Also, two parallel planar surfaces with normal vectors pointing to the opposite directions serve as examples of a pair of 2-congruent features.

- **Complementary:** if there exists $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = \Leftrightarrow\rho_2$ where $\Leftrightarrow\rho_2 = \{(s, \Leftrightarrow v)|(s, v) \in \rho_2\}$; in other words, $\forall (s, v) \in g * \rho_1, \exists (s, \Leftrightarrow v) \in \rho_2$, and $\forall (s, v) \in \rho_2, \exists (s, \Leftrightarrow v) \in g * \rho_1$. See Figure 7 for an example.

It is easy to verify that these relationships are symmetrical relations. Immediately we can prove that this characterization has exhaustively enumerated all the possible cases between a pair of oriented primitive features.

Proposition 3.1.3 Distinct, 1-congruent, 2-congruent *and* complementary *are the only possible relationships between a pair of primitive features.*

Proof :

Given two primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$. Lemma 6.0.19 suggests that either there exists a $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ or no such g exists. Now let us check each case.

Note that any two planar surfaces are complementary of each other, and are either 2-congruent (when the planes intersect or are parallel with their normals pointing in the opposite directions), or 1-congruent (when the planes are parallel with their normals pointing in the same direction).

- If there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$:
  - If $g(S_2) = S_1$ also, then $g(S_1) = g(g(S_2)) = S_2 \Rightarrow g^2 = 1$. Now there are two cases in terms of their orientations (Lemma 6.0.20):

    If $g * \rho_1 = \rho_2$ then $g * \rho_2 = g * g * \rho_1 = \rho_1$. This is the definition of **2-congruent**.

    If $g * \rho_1 = \Leftrightarrow\rho_2$ then , this falls into the definition of **complementary**.
  - If $g(S_2) \neq S_1$ then

    If $g * \rho_1 = \rho_2$ this is the definition of **1-congruent**.

    If $g * \rho_1 = \Leftrightarrow\rho_2$, this is the definition of **complementary**.
- If for any $g \in \mathcal{E}^+, g(S_1) \neq S_2$ (Lemma 6.0.19):

  This is the definition of **distinct**.

□

Corollary 3.1.4 *Except for a pair of planar surface primitive features,* distinct, 1-congruent, 2-congruent *and* complementary *relationships are mutually exclusive relations between a pair of primitive features.*

Proof : As has been shown in the proof of proposition 3.1.3. □

The definition for oriented features allows us to distinguish a feature from its complement which we cannot do for features treated only as sets. In general the relationship between two primitive features can be either distinct, 1-congruent, 2-congruent or complementary, except for a pair of planar surfaces of solids which are always complementary of each other and at the same time can be either 1-congruent or 2-congruent.

When two solids have a surface contact, it is the case that two features which are complementary of each other are brought into coincidence. The following proposition states how the symmetry groups of a pair of complementary features are related to each other.

Proposition 3.1.5 *If features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ are complementary of each other, where $a(S_1) = S_2, a \in \mathcal{E}^+$, and G_1, G_2 are the symmetry groups of F_1, F_2 respectively, then $aG_1a^{-1} = G_2$. In particular, if $S_1 = S_2$ then $G_1 = G_2$* **(the necessary condition for surface contact)**.

Proof :

For all $g = ag_1a^{-1} \in aG_1a^{-1}, g(S_2) = ag_1a^{-1}(S_2) = ag_1(S_1) = a(S_1) = S_2$.

For all $(s, v) \in \rho_2$ by definition of complementary features $(s, \Leftrightarrow v) \in a * \rho_1 = (ag_1a^{-1}a) * \rho_1 = g(a * \rho_1)$, where $g = aga^{-1} = tr \in aG_1a^{-1}$. Thus $(g^{-1}s, r^{-1}v) \in a * \rho_1$. By the definition of complementary features $(g^{-1}s, r^{-1}v) \in \rho_2$. Then $(s, v) \in g * \rho_2$. Therefore $\rho_2 \subseteq g * \rho_2$.

On the other hand, $\forall (gs, rv) \in g * \rho_2, (s, v) \in \rho_2$. By the definition of complementary features $(s, \Leftrightarrow v) \in a * \rho_1$. Then $(gs, \Leftrightarrow rv) \in g(a * \rho_1) = a * \rho_1$. By the definition of complementary features again, $(gs, rv) \in \rho_2$. So $g * \rho_2 \subseteq \rho_2$.

Therefore forall $g \in aG_1a^{-1}, g * \rho_2 = \rho_2$. That is $aG \Leftrightarrow 1a^{-1}$ is a symmetry group for F_2. Hence $aG_1a^{-1} \subseteq G_2$.

Now we need to prove: $G_2 \subseteq aG_1a^{-1}$, i.e. G_2 is a symmetry group of $a(S_1)$.

If $g = tr \in G_2$ then first consider how it acts on the set $g(a(S_1)) = g(S_2) = S_2 = a(S_1)$. Now let us consider how g acts on the orientations. For all $(s, v) \in a * \rho_1, \exists (s, \Leftrightarrow v) \in \rho_2 = g * \rho_2$, then $(g^{-1}s, r^{-1}v) \in \rho_2 \Rightarrow (g^{-1}s, \Leftrightarrow r^{-1}v) \in a * \rho_1 \Rightarrow (s, v) \in g(a * \rho_1)$. So $a * \rho_1 \subseteq g(a * \rho_1)$. On the other hand, $\forall (gs, rv) \in g(a * \rho_1), \exists (s, v) \in a * \rho_1 \Rightarrow (s, \Leftrightarrow v) \in \rho_2 \Rightarrow (gs, \Leftrightarrow rv) \in g * \rho_2 = \rho_2 \Rightarrow (gs, rv) \in a * \rho_1$. So $g(a * \rho_1) \subseteq a * \rho_1$. One can conclude $g(a * \rho_1) = a * \rho_1$. Therefore $G_2 \subseteq aG_1a^{-1}$.

Hence $G_2 = aG_1a^{-1}$. In case $a = 1, G_1 = G_2$. □

The following lemma shows that for any non-planar primitive features, symmetries for the set-features are the symmetries for the oriented features.

Lemma 3.1.6 *For any non-planar primitive feature $F = (S, \rho)$, if there exists an isometry $g = tr$ such that $g(S) = S$ then $g * \rho = \rho$.*

Proof :

By lemma 6.0.20, $g * \rho = \rho$ or $g * \rho = \Leftrightarrow \rho$. To prove by contradiction let us assume that $g * \rho = \Leftrightarrow \rho$. By definition 3.1.2 F is complementary with itself.

Since in Euclidean space a rotation cannot inverse more than two independent vectors simultaneously[5] [12, 4], an oriented surface F has to have less than or equal to two normals in order for all of its normals to be inversed by a rotation. The only such surface, or even a surface which has finite number of normals, is a planar surface.

Thus F is a planar surface, a contradiction. □

This lemma says that a symmetry for the set of points on a surface is a symmetry for the orientation of the surface as well. This could be seen as a justification for treating an oriented

[5]If R is a rotation and $\vec{u}, \vec{v}$ are vectors in Euclidean space, then the vector cross product obeys: $R(\vec{u})XR(\vec{v}) = R(\vec{u}X\vec{v})$

surface as a subset in Euclidean space. Unfortunately, this result does not hold for compound features. Consider the compound feature which is composed of two cylindrical surfaces in case (b) of Figure 2, any transformations which interchange the two surfaces (symmetries of the compound feature) will reverse the orientations at each point of the feature.

3.2 Symmetry Group of Multiple Oriented Surfaces

In the next few propositions we shall explore how the symmetry group of a compound feature is expressed by the symmetry groups of its component primitive features. The first case we consider is when a compound feature F is composed of n pairwise *distinct* features.

Proposition 3.2.7 *Given a compound feature $F = (S, \rho)$ of primitive features $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$ where $F_1, ..., F_n$ are pairwise distinct primitive features with symmetry groups $G_1, ...G_n$ respectively. Then the symmetry group G of F is $G = G_1 \cap ... \cap G_n$.*

Proof :

Let $g \in G$, then $g(S) = S$. Thus $g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n$. Then $g(S_i) \subseteq S_1 \cup ... \cup S_n$.

From Lemma 6.0.19 and the definition of distinct features (Definition 3.1.2) we know that $\forall g \in G, g(S_i) = S_i, i = 1...n$.

By Lemma 3.1.6 we have for all the non-planar primitive features $g * \rho_i = \rho_i$. Since $F_1...F_n$ are pairwise distinct there is at most one planar feature whose orientation has to be mapped to itself.

Therefore $g \in G_i$ for $i = 1, ..., n$. Thus $g \in G_1 \cap ... \cap G_n \Rightarrow G \subseteq G_1 \cap ... \cap G_n$.

For all $g \in G_1 \cap ... \cap G_n, g(S) = g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n = S$ and $g * \rho = g * (\rho_1 \cup ... \cup \rho_n) = g * \rho_1 \cup ... \cup g * \rho_n = \rho_1 \cup ... \cup \rho_n = \rho \Rightarrow g \in G \Rightarrow G_1 \cap ... \cap G_n \subseteq G$.

Therefore $G = G_1 \cap ... \cap G_n$. □

The following definition and three theorems are from [2]. We shall use these in our proofs.

Definition 3.2.8 *Two sets H, K are* **separated** *if*

$$\bar{H} \cap K = H \cap \bar{K} = \emptyset.$$

Theorem 3.2.9 *A set $M \subset X$ is* **connected** *if and only if M is not the union of two nonempty separated sets.*

Theorem 3.2.10 *For sets, connectivity is preserved by surjective mappings.*

Theorem 3.2.11 *If H and K are separated, then every connected subset M of $H \cup K$ lies either in H or in K.*

Next we examine what happens when a compound feature F is composed of a pair of *1-congruent* features. What is the symmetry group of F? Let us first prove one useful lemma:

Lemma 3.2.12 *For any pair of primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ where $S_1 \neq S_2$, if there exists a $g \in \mathcal{E}^+$ such that $g(S_1 \cup S_2) = S_1 \cup S_2$ then $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$.*

Proof :

There are two possibilities for S_1 and S_2:

- $S_1 \cap S_2 = \emptyset$.

  Since $g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = S_1 \cup S_2$, and $g(S_1)$ is a connected subset of $S_1 \cup S_2$ (Theorem 3.2.10), by Theorem 3.2.11 $g(S_1) \subseteq S_1$ or $g(S_1) \subseteq S_2$. If $g(S_1) \subseteq S_1$ then, due to connectivity, $g(S_2) \subseteq S_2$. Since g is a bijection $g(S_1) = S_1, g(S_2) = S_2$. Similarly, $g(S_1) = S_2, g(S_2) = S_1$.

- $S_1 \cap S_2 \neq \emptyset$.

  If there exist open sets $O_1 \subset g(S_1) \cap S_1$ and $O_2 \subset g(S_1) \cap S_2$. Then by Lemma 6.0.19 $g(S_1) = S_1$ and $g(S_1) = S_2$. Thus $S_1 = S_2$, a contradiction. Thus either $g(S_1)$ and S_1 share an open set such that $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$.

Therefore $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$. □

The proposition for finding the symmetry group of a pair of 1-congruent features follows:

Proposition 3.2.13 *Let a compound feature $F = (S, \rho)$ be composed of a pair of primitive features $F_1 = (S_1, \rho_1)$ and $F_2 = (S_2, \rho_2)$ which are 1-congruent of each other. If G_1, G_2 are the symmetry groups of F_1, F_2 respectively, and G is the symmetry group of F then $G = G_1 \cap G_2$.*

Proof :

For all $g \in G, g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2)$ and $g * \rho = g * (\rho_1 \cup \rho_2) = g * \rho_1 \cup g * \rho_2$. By Lemma 3.2.12,

- $g(S_1) = S_1, g(S_2) = S_2$:

  If F_1, F_2 are planar features, they have to be parallel planes with their normals pointing to the same direction, i.e. $\rho = \rho_1 = \rho_2$. Thus $g * \rho = \rho \Rightarrow g * \rho_1 = \rho_1$ and $g * \rho_2 = \rho_2$. For non-planar features $g * \rho_1 = \rho_1, g * \rho_2 = \rho_2$ (Lemma 3.1.6).

- $g(S_1) = S_2, g(S_2) = S_1$:

  By Lemma 6.0.20 If $g * \rho_1 = \rho_2$ then F_1, F_2 are 2-congruent; if $g * \rho_1 = \Leftrightarrow \rho_2$ then F_1, F_2 are complementary; both contradict the fact that F_1, F_2 are 1-congruent.

Then $g \in G_1 \cap G_2$. So we have $G \subseteq G_1 \cap G_2$.

On the other hand, for all $g \in G_1 \cap G_2$, $g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = S_1 \cup S_2 = S$; $g * \rho = g * (\rho_1 \cup \rho_2) = g * \rho_1 \cup g * \rho_2 = \rho_1 \cup \rho_2 = \rho$. Therefore $g \in G \Rightarrow G_1 \cap G_2 \subseteq G$. Thus we conclude $G = G_1 \cap G_2$. □

Similar result can be obtained for a pair of complementary features. Lastly we consider the symmetry group of a compound feature F which is composed of a pair of *2-congruent* features.

Proposition 3.2.14 *Let a compound feature $F = (S, \rho)$ be composed of a pair of primitive features F_1 and F_2 which are 2-congruent of each other via g_c (Definition 3.1.2). If $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ have symmetry groups G_1, G_2 respectively, and G is the symmetry group of F then $G =< g_c > (G_1 \cap G_2)$ where $< g_c >$ denotes the subgroup of $\mathcal{E}^+$ generated by g_c.*

Proof :

If $g \in G$ then by Lemma 3.2.12 either

- $g(S_1) = S_1$ and $g(S_2) = S_2$:

  By Lemma 3.1.6, taking planar feature case into consideration also, $g * \rho_1 = \rho_1, g * \rho_2 = \rho_2$. Thus $g \in G_1$ and $g \in G_2 \Rightarrow g \in G_1 \cap G_2$; or

- or $g(S_1) = S_2$ and $g(S_2) = S_1 \Rightarrow g^2 = 1$:

  g can be written as $g = g_c g_c^{-1} g$. Let $g_0 = g_c^{-1} g$. $g_0(S_1) = g_c^{-1} g(S_1) = g_c^{-1}(S_2) = S_1, g_0 * \rho_1 = (g_c^{-1} g) * \rho_1 = g_c^{-1} * \rho_2 = g_c * \rho_2 = \rho_1$ (Lemma 6.0.20).

  Therefore $g_0 \in G_1$. Similarly we can prove $g_0 \in G_2$. Thus $g_0 \in G_1 \cap G_2 \Rightarrow g \in < g_c > (G_1 \cap G_2) \Rightarrow G \subseteq < g_c > (G_1 \cap G_2)$;

Therefore $G \subseteq < g_c > (G_1 \cap G_2)$.

On the other hand, if $g \in < g_c > (G_1 \cap G_2)$ then $g = g' g_{12}$ where $g' \in < g_c >$ and $g_{12} \in G_1 \cap G_2$. Then $g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = g' g_{12}(S_1) \cup g' g_{12}(S_2) = g'(S_1) \cup g'(S_2)$. By lemma 3.2.12, either $g'(S_1) \cup g'(S_2) = S_1 \cup S_2 = S$ or $g'(S_1) \cup g'(S_2) = S_2 \cup S_1 = S$. For orientations $g * \rho = g * (\rho_1 \cup \rho_2) = g' g_{12} * \rho_1 \cup g' g_{12} * \rho_2 = g' * \rho_1 \cup g' * \rho_2$. Since $g' \in < g_c >$, by definition of 2-congruent (Definition 3.1.2) either $g' * \rho_1 \cup g' * \rho_2 = \rho_1 \cup \rho_2 = \rho$ or $g' * \rho_1 \cup g' * \rho_2 = \rho_2 \cup \rho_1 = \rho$. Therefore $g \in G \Rightarrow < g_c > (G_1 \cap G_2) \subseteq G$.

Thus we conclude $G =< g_c > (G_1 \cap G_2)$. □

In general, the symmetry group G of a compound feature F can be found from the intersection of the symmetry groups G_i of its primitive features. The only exception is the case where the mapping which flips pairwise 2-congruent features in F also contribute to G, these are usually new symmetries do not exist in any G_i and the new group they generated are discrete groups. With this proposition we end this section where propositions are proved for the symmetry groups of all the possible pairs of the oriented primitive features.

As one can observe from the proved results for surface contact, the intersection of symmetry groups of the primitive features is one of the crucial operations in determining the relative motions of contacting solids. We face two computational problems:

1. How to denote symmetry groups, which can be finite, infinite, discrete or continuous, on computers?

2. How to do intersections of subgroups of $\mathcal{E}^+$ on computers efficiently?

We have implemented an efficient group denotation and intersection algorithm for computing compound feature symmetry groups [8]. The symmetry group of each surface is obtained by a straightforward mapping from the boundary (surface) file of a solid to their respective canonical symmetry groups.

4 Applications

As one can observe from the proved results for surface contact, the intersection of symmetry groups of the primitive features is one of the crucial operations in determining the relative motions of contacting solids. We face two computational problems in practice:

1. How to denote symmetry groups which can be finite, infinite, discrete or continuous, on computers?

2. How to intersect subgroups of $\mathcal{E}^+$ on computers efficiently?

We have successfully implemented an efficient group intersection algorithm using geometric invariants denotation of the groups [8]. The basic symmetry group of each surface of a solid is obtained by a straightforward mapping from the boundary (surface) file of the solid to their respective canonical symmetry groups.

As an example of assembly specification using symmetry groups, see Figures 8 and 9 for a five-part gearbox. The representation of the assembly is shown in Figure 10, where $L_i, i = 1..4$

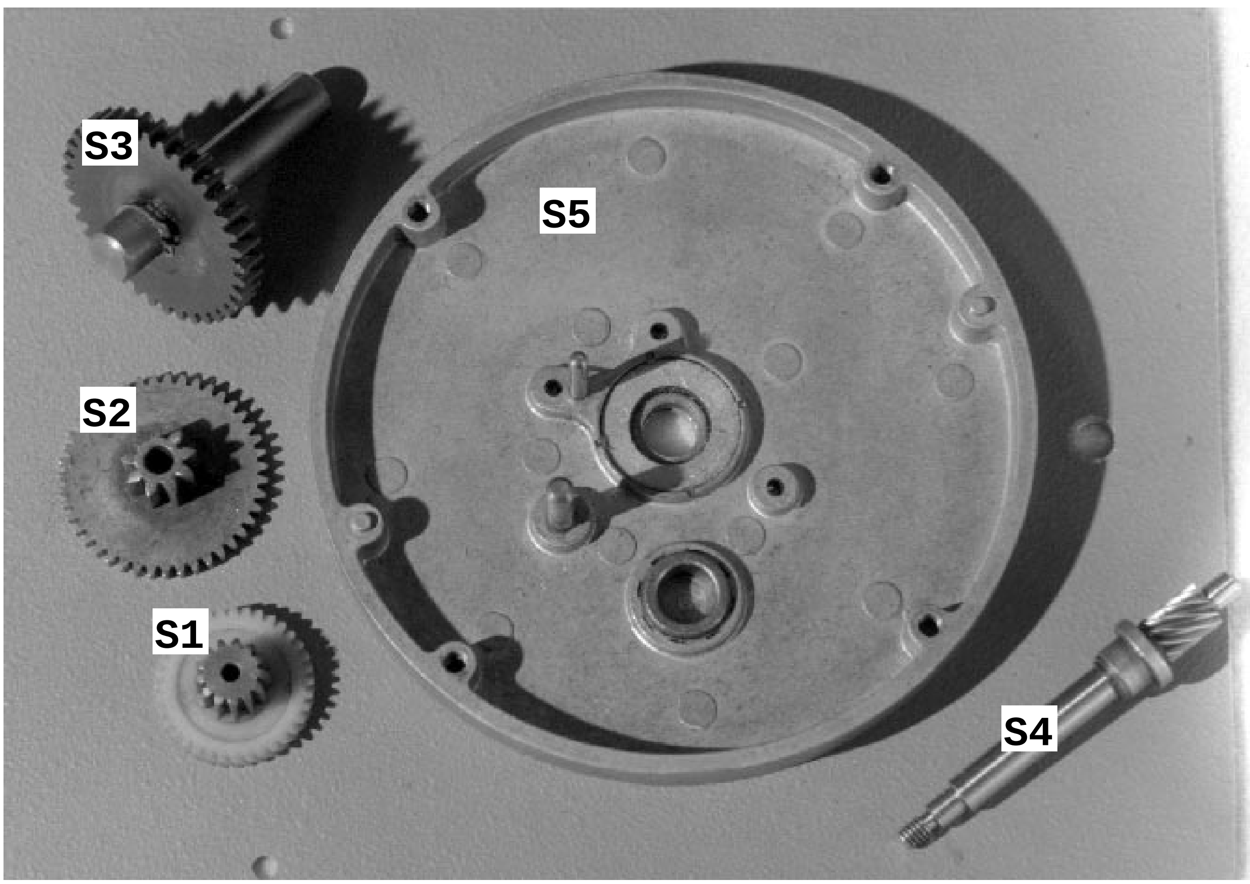

Figure 8: A five-part Gearbox

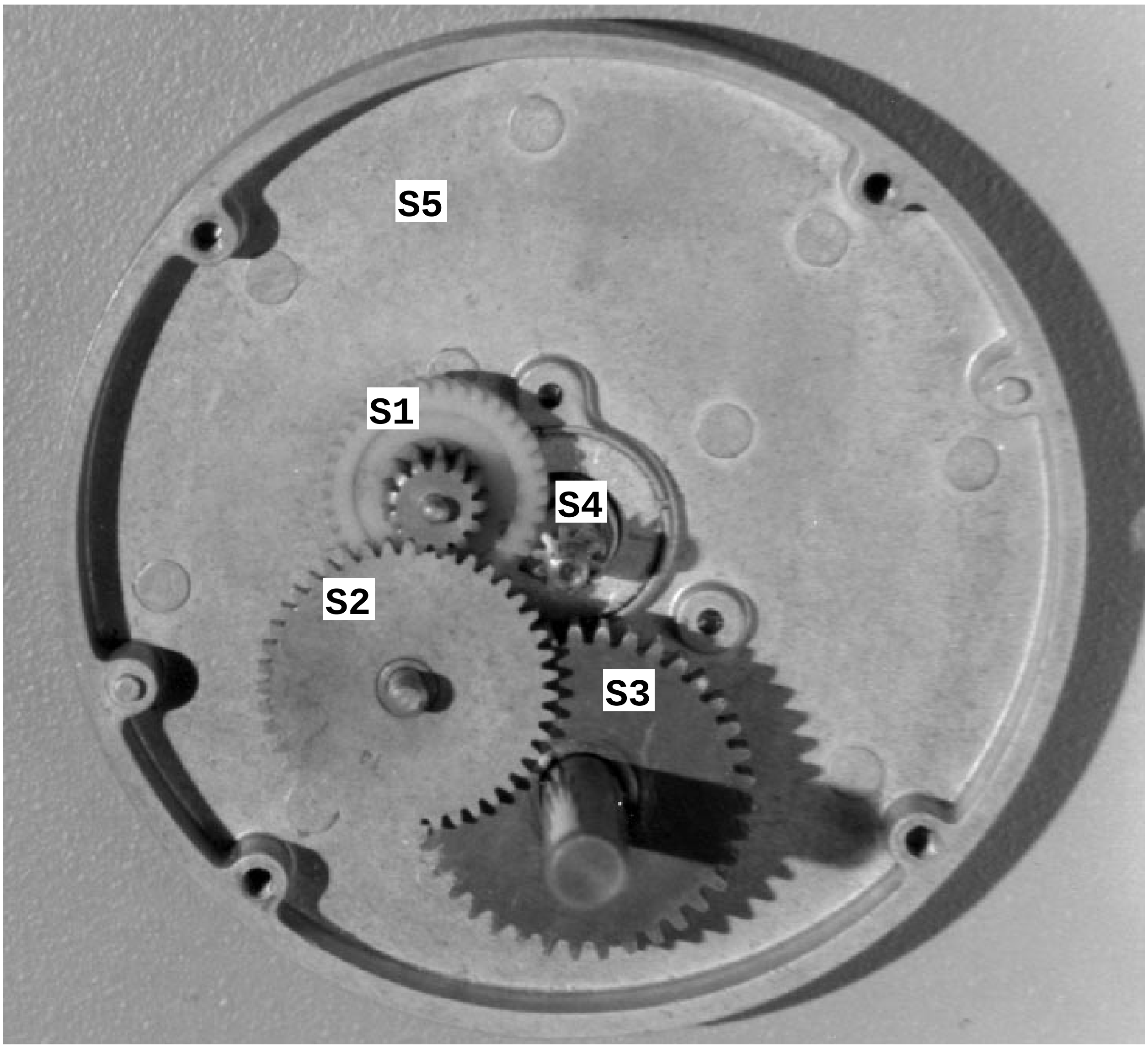

Figure 9: The top view of the gearbox when it is assembled.

is the symmetry group of the contacting compound feature between solids S_i and S_5. $L_i = a_i SO(2) a_i^{-1}, i = 1..4$, $SO(2)$ is a one degree rotation group resulted from the intersection of the symmetry group of a plane with that of a cylinder (the compound feature composed of two surfaces of the shaft of a gear). $L_{ij} = L_i L_j = a_{ij} SO(2) b_{ij} SO(2) c_{ij}, i, j = 1..4$ indicate the relative positions between gears (non-surface contact) are simply determined by some translations a_{ij}, b_{ij}, c_{ij}, where the relative gear pitch ratio is also embedded, and a rotation in $SO(2)$. This representation of the gearbox (Figure 10) specifies precisely the articulated gearbox assembly.

Figure 11 from [17] shows a nonlinearizable assembly. Using our representation, one can immediately determine it is a nonlinearizable assembly by computing the symmetry group of the contacting surfaces for each individual part under any possible motion. The result is an identity group, meaning no existing relative motions between the part and the rest of the assembly that can separate the part. A disassemblable subassembly can be identified by computing the symmetry group of a compound feature composed of contacting surfaces from 2 or 3 parts under a disassembly motion, If the resulting group remains as an identity group then the subassembly is not movable. In this example, the resulting group of contacting surfaces for a two-part subassembly, such as $S_1 \cup S_2$, is a T_1 group indicating a one dimensional translation which can separate the assembly.

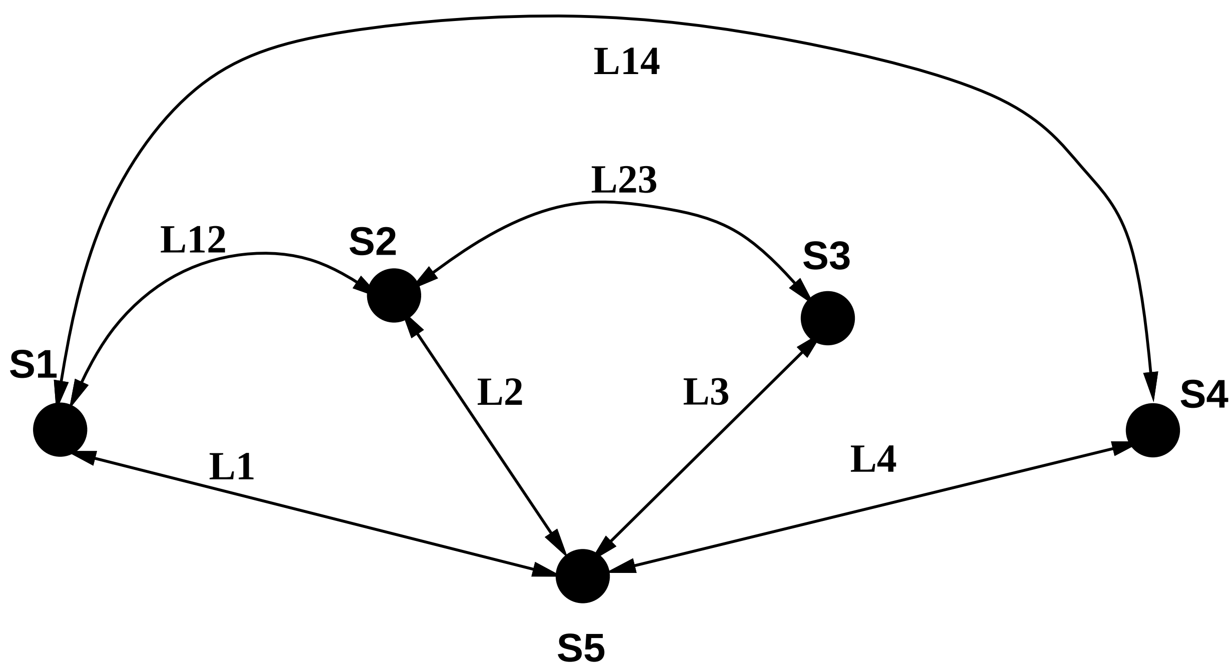

Figure 10: Representation of the gearbox assembly in terms of contacting compound feature symmetry groups, where $L_i = a_i SO(2) a_i^{-1}, i = 1..4$ and $L_{ij} = L_i L_j = a_{ij} SO(2) a_{ij}^{-1}, i, j = 1..4$

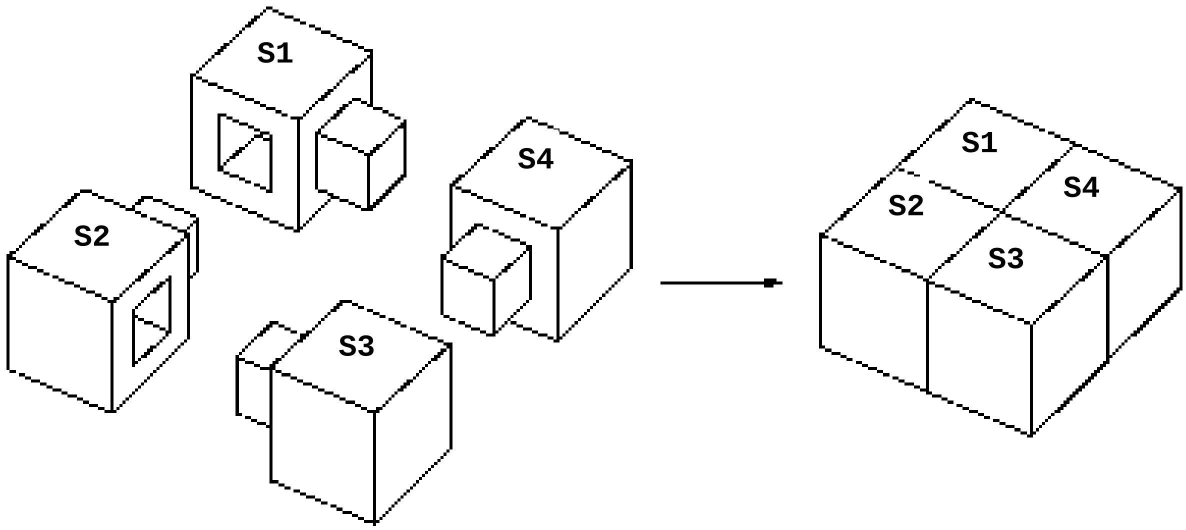

Figure 11: A four-part nonlinearizable assembly from [13].

5 Summary and Discussion

In summary, by a *computational representation* of an assembly we mean that the representation of the assembly can be directly used to compute, for example:

1. relative positions of its parts in the final assembly configuration
2. the type and range of motion of any subset of parts in the assembly
3. separation of subassemblies
4. compliant motion for (dis)assembly process

Here are some questions we seek the answers for:

1. Given two solids S_1, S_2, what is the relative location of the two under n surface contact (n primitive features from each side, Figure 12)?
2. Given two solids S_1, S_2, what is the relative location of the two under n general contact?
3. Given m solids in a chaining general contact (Figure 13), what is the relative location of the mth solid with respect to the first solid?

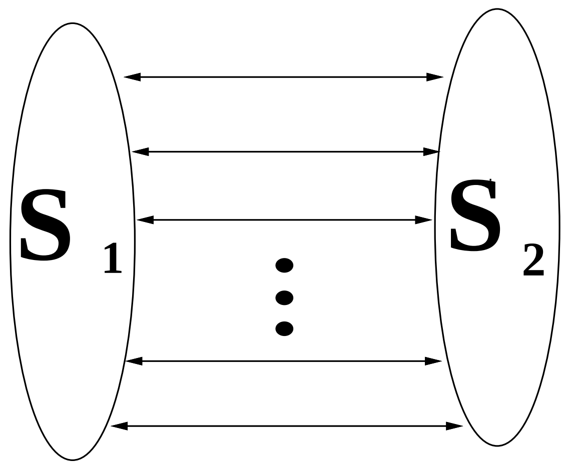

Figure 12: Solids S_1 and S_2 have n contacts

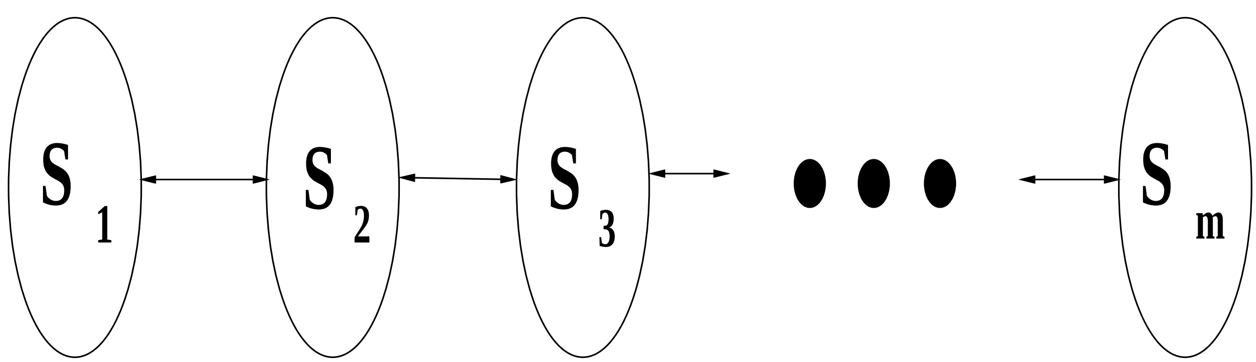

Figure 13: Solids $S_1, S_2, \ldots S_m$ form a chain

From our reasoning so far we can express the relative locations in each case above using the symmetry group of the contacting surfaces:

1. Two solids have n surface contact, the relative position of solid 2 with respect to solid 1:

$$l_1^{-1}l_2 \in f_1 G f_2^{-1} \tag{3}$$

where G is the symmetry group of the compound feature composed of *all* the contact primitive features of S_1 or S_2. If F is composed of n pairwise distinct features F_i, then $G = \cap G_i$, where G_i is the symmetry group of F_i. What about other cases?

2. Two solids have n general contact, the relative position of solid 2 with respect to solid 1:

$$l_1^{-1}l_2 \in f_{11}G_{11}\sigma_1 G_{21} f_{21}^{-1} \cap f_{12}G_{12}\sigma_2 G_{22} f_{22}^{-1} \cap \ldots \tag{4}$$
$$\cap f_{1n}G_{1n}\sigma_n G_{2n} f_{2n}^{-1}$$

where G_{ij} is the symmetry group of primitive feature j of S_i and f_{ij} is its feature coordinates.

3. m solids have a chaining general contact, the relative location of solid m with respect to solid 1:

$$l_1^{-1}l_m \in f_1 G_{12}\sigma_1 G_{21} f_{21}^{-1} f_2 G_{23}\sigma_2 G_{32} f_{32}^{-1} \ldots \tag{5}$$
$$f_{m-1}G_{(m-1)m}\sigma_{m-1}G_{m(m-1)}f_{m(m-1)}^{-1}$$

where G_{ij} is the symmetry group of the surface on solid i in contact with solid j.

In this article we have carefully examined the representation and computation aspects of oriented surfaces. Special attention is given to the characterization of symmetry groups for contacting surfaces among solids. The next step is to further study those compound features with more complicated inner structures. For example, one may define a concept of n-congruence on n features $F_1 \ldots F_n$ as requiring that there exists $g \in \mathcal{E}^+$ such that $g(F_i) = F_{(i \bmod n)+1}$; this is a natural extension of 2-congruence. Such congruences will give rise to new symmetries of the compound feature. However Proposition 3.2.14 is not trivially generalized to such a proposition:

Proposition 5.0.15 *Given a compound feature $F = (S, \rho)$ of primitive features $F_1 = (S_1, \rho_1), \ldots, F_n = (S_n, \rho_n)$ with symmetry groups $G_1, \ldots G_n$ respectively, the symmetry group G of F is*

$$G = <\{g_{ij}\}> (G_1 \cap \ldots \cap G_n)$$

where $\{g_{ij}\}$ is a set of isometries $g_{ij} \in \mathcal{E}^+$, each of which is associated with a pair of 2-congruent primitive features F_i, F_j in F such that F_i, F_j are 2-congruent via g_{ij}; $<\{g_{ij}\}>$ denotes the group generated by all such g_{ij}s.

Our new results on oriented surfaces has laid out a realistic and precise group theoretic framework for characterizing surfaces of solids and capture the very nature of surface contact — the state of being complementary. Under this formalization surface contact can be treated conceptually effectively and computationally efficiently. In this paper we have generalized this framework and applied it to provide a concise, complete and computational representation for rigid and articulated assembly. Further work is needed for a computational treatment of group products as what have been implemented for group intersections in [7, 8].

6 Appendix

Here we demonstrate a set of proofs for some basic lemmas and propositions used in the previous text.

In the next two lemmas we prove the associativity of isometries when they act on the relation ρ.

Lemma 6.0.16 *For all* $g_1, g_2 \in \mathcal{E}^+, (g_1 g_2) * \rho = g_1 * (g_2 * \rho)$.

Proof:

Let $g_1 = t_1 r_2, g_2 = t_2 r_2$ where $t_1, t_2 \in \mathbf{T}^3, r_1, r_2 \in SO(3)$. Since $g_1 g_2 = t_1 r_1 t_2 r_2 = t_1 t' r_1 r_2$ ($\mathbf{T}^3$ is a normal subgroup of $\mathcal{E}^+$), for all $(s, v) \in \rho, (g_1 g_2 s, r_1 r_2 v) \in (g_1 g_2) * \rho$. On the other hand, for all $(s, v) \in \rho, (g_2 s, r_2 v) \in g_2 * \rho$ and $(g_1 g_2 s, r_1 r_2 v) \in g_1 * (g_2 * \rho)$. Therefore, $(g_1 g_2) * \rho = g_1 * (g_2 * \rho)$.
□

Lemma 6.0.17 *For all* $g_1, g_2, g_3 \in \mathcal{E}^+, (g_1 g_2) * (g_3 * \rho) = g_1 * ((g_2 g_3) * \rho)$.

Proof:

Given the commutativity diagram and the closeness property of $\mathcal{E}^+$ we have
$(g_1 g_2) * (g_3 * \rho) = (g_1 g_2 g_3) * \rho = g_1 * ((g_2 g_3) * \rho)$. □

Proposition 6.0.18 *The proper symmetries of a set* $S \subseteq \Re^3$ *form a subgroup of* $\mathcal{E}^+$.

Proof :

Let G denote the set of the symmetries of $S \subset \Re^3$. Obviously, $1(S) = S$, so $1 \in G$. If $g \in G$ then $g(S) = S$, multiplying by g^{-1} we have $g^{-1} g(S) = g^{-1}(S)$ therefore $g^{-1}(S) = S$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2)(S) = g_1(g_2(S)) = g_1(S) = S$ therefore $g_1 g_2 \in G$. By the definition of a subgroup G is a subgroup of $\mathcal{E}^+$. □

Lemma 6.0.19 *Given two primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$. If there exists an open set O such that $O \subset S_1 \cap S_2$ then $S_1 = S_2$. In another words, if S_1, S_2 are locally identical then they are identical globally.*

Proof :

Analytic functions have the property that if they are locally identical then they are globally identical [1]. In the definition of primitive features (Definition 2.2.2), S_1, S_2 are defined by irreducible algebraic functions, which form a subset of the analytic functions, and thus they inherit the property. Therefore if S_1 and S_2 share an open set then $S_1 = S_2$. $\square$

Lemma 6.0.20 *Given two primitive features $F_1 = (S_1, \rho_1)$, and $F_2 = (S_2, \rho_2)$. If there exists $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ then either $g * \rho_1 = \rho_2$ or $g * \rho_1 = \Leftrightarrow \rho_2$.*

Proof :

By Definition 2.2.2, any point s on a primitive feature has either

- a non-singular point with a unique tangent plane:

  there are two possible antipodal normals for each plane, say $v, \Leftrightarrow v$. By the definition of a primitive feature either $(s, v) = (s_2, v_2) \in \rho_2$ or $(s, \Leftrightarrow v) = (s_2, v_2) \in \rho_2$. Since ρ_1, ρ_2 are continuous mappings and isometry g does not change their continuity, for $(s_1, v_1) \in \rho_1$,

  - if $rv_1 = v_2$ then $g * \rho_1 = \rho_2$,
  - if $rv_1 = \Leftrightarrow v_2$ then $g * \rho_1 = \Leftrightarrow \rho_2$; or

- a singular point with an infinite number of "tangent planes":

  there is an infinite set of normals which are determined by the neighborhoods of the singular point s. Each of such neighborhoods is composed of non-singular points, Thus the above argument also applies.

$\square$

References

[1] L.V. Ahlfors. *Complex Analysis*. McGraw-Hill, New York, 1979.

[2] F.H. Croom. *Basic Concepts of Algebraic Topology*. Springer-Verlag, New York, 1978.

[3] M.P. Do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice Hall, New Jersey, 1976.

[4] M. Hamermesh. *Group Theory and its application to physical problems*. Addison-Wesley, Mass, 1962.

[5] L.S. Homem de Mello. *Task Sequence Planning for Robotic Assembly*. PhD thesis, Carnegie Mellon University, 1989.

[6] C. Laugier. Planning fine motion strategies by reasoning in the contact space. In *IEEE International Conference on Robotics and Automation*, pages 653–661, Washington, DC, 1989. IEEE Computer Society Press.

[7] Y. Liu. *Symmetry Groups in Robotic Assembly Planning*. PhD thesis, University of Massachusetts, Amherst, MA., September 1990.

[8] Y. Liu. A Geometric Approach for Denoting and Intersecting TR Subgroups of the Euclidean Group. *DIMACS Technical Report, Rutgers University*, 93-82:1–52, 1993.

[9] Y. Liu and R. Popplestone. A Group Theoretical Formalization of Surface Contact. *International Journal of Robotics Research*, 13(2):148 – 161, April 1994.

[10] Y. Liu and R.J. Popplestone. Assembly planning from solid models. In *IEEE International Conference on Robotics and Automation*, Washington, DC, 1989. IEEE Computer Society Press.

[11] T. Lozano-Pérez, J.L. Jones, E. Mazer, P.A. O'Donnell, W.E.L. Grimson, P. Tournassoud, and A. Lanusse. Handey: A robot system that recognizes, plans, and manipulates. In *IEEE International Conference on Robotics and Automation*, pages 843–849, Washington, DC, March 1987. IEEE Computer Society Press.

[12] W. Miller Jr. *Symmetry Groups and Their Applications*. Academic Press, New York, 1972.

[13] David Montana. The kinematics of contact and grasp. *The International Journal of Robotics Research*, 7(3):17–32, June 1988.

[14] R.J. Popplestone, A.P. Ambler, and I. Bellos. An interpreter for a language for describing assemblies. *Artificial Intelligence*, 14(1):79–107, 1980.

[15] F. Thomas and C. Torras. A group-theorectic approach to the computation of symbolic part relations. *IEEE Journal of Robotics and Automation*, 4(6):622–634, December 1988.

[16] F. Thomas and C. Torras. Inferring feasible assemblies from spatial constraints. *IEEE Transactions on Robotics and Automation*, 8(2):228,239, April 1992.

[17] Randall Wilson and Jean-Claude Latombe. Geometric reasoning about mechanical assembly. *Artificial Intelligence*, 71(2), December 1994.

[18] J.D. Wolter. *On the Automatic Generation of Plans for Mechanical Assembly*. PhD thesis, University of Michigan, 1988.