Geometric Solution of Shape from Shading Problem

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Abstract

A new geometrical approach in solving the shape from shading problem of Lambertian model is discussed. It is shown, through the global and local geometrical analysis, that the problem is bound by two local constraints. The C^1 continuous analytic solution is derived. The solution has two degrees of freedom due to the non-linearity of the problem. This shows the existence and the non-uniqueness of the solution.

1 Introduction

The Shape From Shading (SFS) problem of Lambertian reflectance model[13] is a non-linear Partial Differential Equation (PDE) of first order such that

$$F(X,Y,Z,p,q) = 0, \quad p \equiv Z_X, \quad q \equiv Z_Y \qquad (1)$$

requiring the formal assumption of $F_p^2 + F_q^2 \neq 0$, and at least C^2 continuity of $F(\cdot)$ at each point and in some neighborhood of this point for the existence of solution[4, 7, 30, 26, 25]. This problem is interpreted geometrically, in X,Y,Z space, as finding the C^2 surface $Z(X,Y)$, contacted by local tangent patch which has the normal vector $(p,q,$-1) at a point in $Z(X,Y)$.

This PDE problem is widely studied in mathematics, physics, and computer vision. One of the most traditional approach is the Characteristic Strip Expansion Method (CSEM) in the area of differential geometry[27, 29, 14] and PDE[4, 7, 10, 16, 30, 31]. Recently, viscosity solution[2, 3] and/or Level Curve Propagating Method (LCPM)[24, 25, 28] have been introduced for overcoming the defects of CSEM.

Typically, in computer vision, there have been two types of numerical approaches, optimizing approach and geometrical one, employed in the solution of the SFS problem(see [12, 32] for typical approaches). Numerous optimizing techniques for overcoming ill-constrained nature of the problem have been tried by iteratively minimizing a cost function which basically describes the differences between the model and observed image. The main problem of these approaches is that these usually require a lots of iterations without guarantee of convergence. Geometrical approaches, a way of directly solving the problem, have been started from CSEM[11], and an analysis about properties of characteristic strip has been studied by [22]. Recently, the number of iterations have been considerably reduced in addition to guarantee of convergence by introducing stable approaches based on viscosity solution[26, 21, 6, 9] and/or LCPM[1, 25, 8, 18, 28]. All these approaches are basically similar in the global propagation and/or iteration with a given initial condition which is a non-characteristic curve or a singular point.

Nevertheless, we believe that the study about uniqueness and existence of locally analytic solution of SFS problem have not been fully proceeded until now regardless of recent contributions[22, 23, 5, 6, 20]. The topic about existence and uniqueness of the SFS problem will be discussed based on a typical geometrical interpretation of PDE by assuming known position of light source, normalized albedo, and the orthographic projection. It will be shown by this discussion that the solution of this problem is locally analytical because it is exactly bound by two orthogonal constraints.

The concept of global solution and the existence of local solution will be discussed in Sect. 2, and the C^1 analytic solution will be derived in Sect. 3. Accuracy of reconstruction will be shown in 4 through simulations, being immediately followed by discussions in Sect. 5.

2 Concept of geometrical solution

Let $\vec{n}$ defined by $(p,q,$-1) be the normal vector of a point $(X,Y,Z(X,Y))$ existing on C^2 continuous and compact surface and $\vec{L}$ defined by $(p_l,q_l,$-1) the position vector of a light source. Then the Lambertian reflectance $R_o(X,Y)$ is described as

$$\begin{aligned} R_o(X,Y) &= \vec{L}\cdot\vec{n} = \cos\theta \\ &= \frac{1+p_lp+q_lq}{\sqrt{1+p_l^2+q_l^2}\sqrt{1+p^2+q^2}}, \end{aligned} \qquad (2)$$

where the angle of intersection θ defined by $\cos^{-1}(\vec{L}\cdot\vec{n})$ denotes the angle between $\vec{L}$ and $\vec{n}$. The purpose of SFS problem is to recover the sur-

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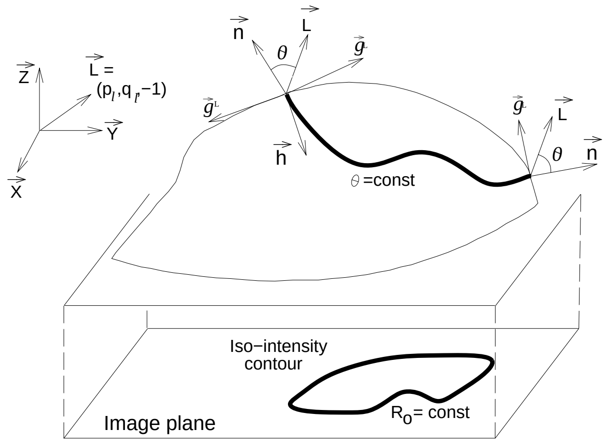


Figure 1: A iso-intensity curve existing on C^2 continuous surface, and corresponding iso-intensity contour existing on image plane.

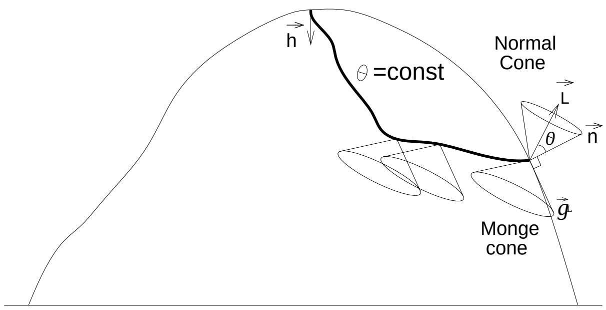


Figure 2: A family of cones existing on a level curve. Upper cone is the solution of $\vec{n}$. Lower one, the solution of $\vec{g^L}$, is called the Monge cone and contacts tangent surface.

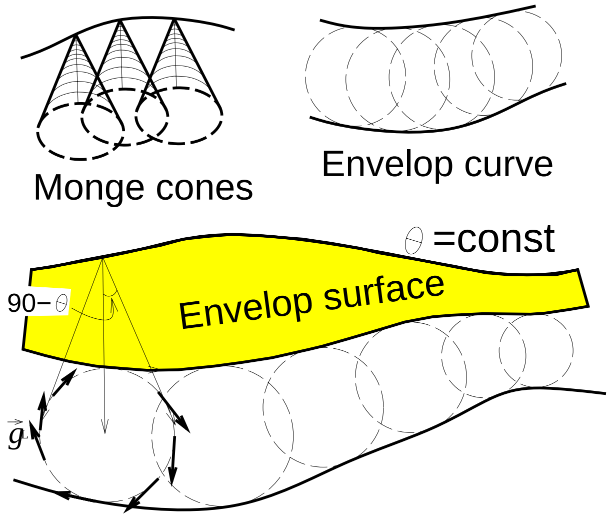


Figure 3: A family of the Monge cones on a level curve (upper left) and that of envelop curves contacted by circles (upper right) constitutes a conoid surface on the level curve.

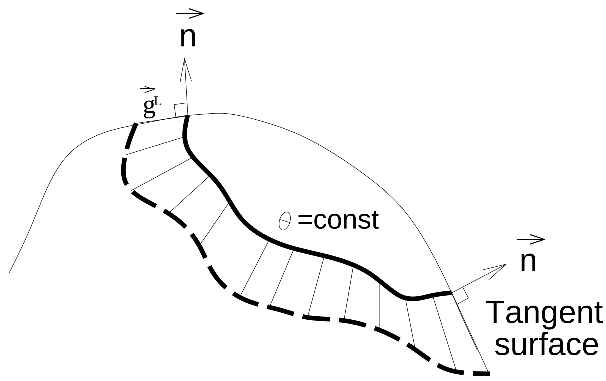


Figure 4: One of tangent surfaces generated by enveloping conoid surface.

face $Z(X,Y)$ or the orientation of tangent plane of $Z(X,Y)$ using $R_o(X,Y)$ and given constraint(s).

2.1 Concept of global solution

Level curve propagation methods[1, 25, 18], which are based on the concept of curvature-dependent speed of propagation of level curves starting from an initial condition, have been successfully applied in finding the solution of SFS problem. The level curve used in this methods corresponds to a spatial curve which is formed by the intersection between the object surface $Z(X,Y)$ and a plane having equal height with respect to the light source.

Similarly, it is possible to introduce another set of level curves, which can recover the object surface compactly, having equal value of intersection angle θ with respect to the light source as is shown in Fig. 1. Because each level curve is parameterized by a constant value $\theta \equiv \ cos^{-1}(\vec{L} \cdot \vec{n})$, this C^2 curve has a constant reflectance value, $R_o \equiv \ \cos(\theta)$, on the object surface and on the observed image which is the orthogonal projection of the surface. At each point (X,Y,Z) in surface, it is possible to define trihedral vectors $\vec{n}, \vec{h}$, and $\vec{g^L}$, where $\vec{n}$ denotes the normal vector of the surface, $\vec{h}$ a tangent vector along the level curve, and $\vec{g^L}$ another tangent vector which is orthogonal to $\vec{n}$ and $\vec{h}$. Then, $\vec{g^L}$ is in the direction of steepest descent/ascent variation of depth relative to the light source. Regardless of ways of reconstructing the surface, the angle θ should be one of the constraints of the SFS problem. So at a point, (X,Y,Z), general solution of $\vec{n}$ determined by $R_o(X,Y)$ and $\vec{L}$ lies on the ambiguous cone[18], as is shown in Fig. 2, which is the envelop of local tangent patch. Consequently, general solution of $\vec{g^L}$ becomes another cone parameterized by the angle θ, and which is called the Monge cone[4, 10, 16].

The general solution of an one-parameter family of the Monge cones existing on a C^2 continuous level curve becomes a conoid surface, as is shown in Fig. 3, which corresponds to the tangent surface contacting the level curve[4, 16]. This is a trivial

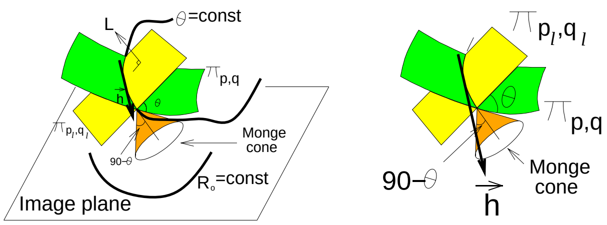


Figure 5: Local tangent patches $\pi_{p,q}$ existing on a level curve have angle of intersection $\pm\theta$ with respect to the plane π_{p_l,q_l} which is perpendicular to $\vec{L}$.

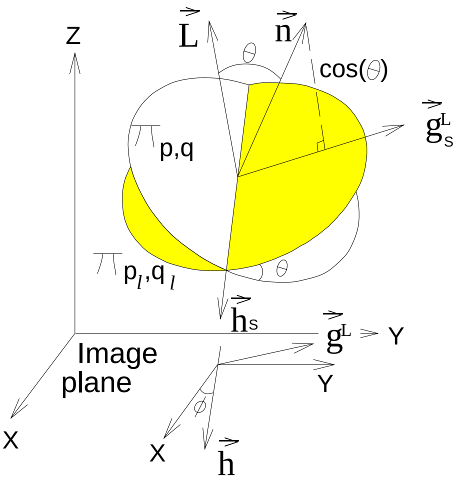


Figure 6: Two constraints, angles θ and ϕ, constrains the SFS problem orthogonally.

conclusion of a theorem in differential geometry; the envelop of envelop of a solution surface is the solution surface caused by i) a cone is composed of stacked circles, and ii) a family of circles having their center on a curve is contacted by two envelop curves. Therefore the solution of tangent surface contacting a level curve becomes a paired family of enveloping conoids having angle π-θ of separation with respect to the generating cylinder enveloped by the light source vector as is shown in Fig. 4. Since the level curve compactly covers the object surface, the tangent surfaces contacted by level curves compactly covers the surface and becomes the solution of surface after integration. This proves the existence of global solution of the SFS problem.

Typically two local tangent surfaces can be obtained because of the property of enveloping surface. The integration of one set of tangent surfaces generally gives the unique solution of continuous surface $Z(X, Y)$ when there is no topological change between convexity and concavity of surface with respect to the light source at a local point. It is distinct that two solutions are possible for the problem, due to the non-linearity of the problem, as already discussed in [1, 22]. Generally, the existence of the solution at singular points and in occluding boundary is not guaranteed by the global approach[26, 25].

2.2 Existence of local solution

The existence of global solution comes from the compact existence of the Monge cones on a level curve. This means that the SFS problem is bound by two constraints, i.e. the intersection angle θ and C^2 continuity which are the formal assumption of the problem required for the existence of the solution[4, 7, 30]. A Local tangent patch which is a part of a conoid surface exists at any point of a level curve due to the compactness of the curve. These can be approximated into the local tangent plane $\pi_{p,q}$. The plane $\pi_{p,q}$ has the angle of intersection $\pm\theta$ with respect to the plane π_{p_l,q_l} as is shown in Fig. 5. This is a local interpretation of the global concept of solution. Actually, this is a trivial result of elementary geometry because

$$\vec{L} \cdot \vec{n} \equiv \cos\theta \equiv \pi_{p_l,q_l} \cap \pi_{p,q} \qquad (3)$$

when $\pi_{p_l,q_l} \perp \vec{L}$ and $\pi_{p,q} \perp \vec{n}$.

Another constraint is clearly the direction of a level curve which can be approximated as the line of intersection by two planes π_{p_l,q_l} and $\pi_{p,q}$. This direction described by $\vec{h}$ should be determined from the observed image which is a orthogonal projection of object surface. The direction of $\vec{h}$ first fixes the axis of rotation of a tangent patch; then this patch is rotated with respect to the plane π_{p_l,q_l} by this axis to the amount of angle $\pm\theta$. This rotation provides the existence of a local solution having two degrees of freedom in the SFS problem.

Although the patch $\pi_{p,q}$ is constrained by the plane π_{p_l,q_l}, it doesn't mean that the corresponding normal vector $\vec{n}$ is variant with respect to the variation of $\vec{L}$. The variants are both the angle θ and the direction of $\vec{g^L}$, i.e. the level curve, because the geometrical relation between $\vec{L}$ and $\vec{n}$ determines the level curves in the process of shading. The vector $\vec{n}$ is invariant, and consequently the integrated surface is invariant as well; and which satisfies the conservation law of Hamilton-Jacobi equations[2, 25].

3 Derivation of analytic solution

The line of intersection generated by the intersection of two planes, π_{p_l,q_l} and $\pi_{p,q}$, can be represented as another angle ϕ defined by the direction of the vector $\vec{h_G}$, the orthogonal projection of $\vec{h}$, as is shown in Fig. 6. The existence of this angle is supported by the existence and intersection of two planes, or the existence of the SFS problem itself; and this is another manifestation of the continuity because the tangent surface is defined in continuous domain. Two constraints represented by angles θ and ϕ are independent each other and satisfy eq. (2) and formal assumption simultaneously. Since these two constrain the problem orthogonally, it is possible to calculate the normal vector at any point of image without considering global property of the object surface provided that ϕ can be determined from the observed image. The way of determining ϕ will be discussed in another paper[17] because it is another topic which should be comprehensively approached.

The analytic solution of the SFS problem then can be derived using two constraints θ and ϕ.

The equation of a line defined by the angle ϕ is described as

$$Y \cos\phi = X \sin\phi, \tag{4}$$

and the equations of two planes π_{p_l,q_l} and $\pi_{p,q}$ are, respectively, described as

$$\pi_{p_l,q_l} = -p_l X - q_l Y + Z_1(X,Y) = 0, \tag{5}$$

$$\pi_{p,q} = -pX - qY + Z_2(X,Y) = 0. \tag{6}$$

The subtraction of eq. (6) from eq. (5) becomes

$$(p - p_l)X + (q - q_l)Y = 0 \tag{7}$$

because $Z_1(X,Y) = Z_2(X,Y)$ at the center of each pixel.

The general linear relation between p and q is then derived from eqs. (4) and (7)

$$(p - p_l)\cos\phi = -(q - q_l)\sin\phi. \tag{8}$$

This indicate that p and q are linearly connected because these are constrained by the equation of plane, i.e. by eq. (6).

The solution then can be derived.

1) General case when $\phi \neq \frac{N}{2}\pi$ (N : integer)

After rewriting eq. (8) into

$$p = -q\tan\phi + (p_l + q_l\tan\phi), \tag{9}$$

and after assigning $R \equiv R_o\sqrt{1+p_l^2+q_l^2}$, new form of eq. (2) becomes

$$R = \frac{1 + p_l p + q_l q}{\sqrt{1+p^2+q^2}}. \tag{10}$$

By substituting eq. (9) into eq. (10) for the elimination of p and by solving the resultant equation with respect to q, the following set of analytic solutions having two degrees of freedom is derived

$$\begin{aligned} q &= \frac{B \pm R\sqrt{D}}{A}, \\ p &= -q\tan\phi + (p_l + q_l\tan\phi), \end{aligned} \tag{11}$$

where

$$A \equiv R^2\sec^2\phi - (q_l - p_l\tan\phi)^2,$$

$$B \equiv (p_l + q_l\tan\phi)\left(R^2\tan\phi + p_l(q_l - p_l\tan\phi)\right) + (q_l - p_l\tan\phi),$$

$$D \equiv \left(\sec^2\phi + (p_l + q_l\tan\phi)^2\right)(1 - R^2) + (p_l + q_l\tan\phi)^2(1 + p_l^2 + q_l^2) + (q_l - p_l\tan\phi)^2.$$

Let (p, q_+) be the positive solution represented by $q_+ = (B + R\sqrt{D})/A$, and (p, q_-) be another one by negative sign; then (p, q_+) and (p, q_-) correspond to convex and concave solution of surface when the over-head light source is used.

2) Special cases

a) When $\phi = \frac{2N}{2}\pi$ (N : integer)

Equation (8) is satisfied only when $p = p_l$ because $\cos\phi = \pm 1$ and $\sin\phi = 0$. The solution becomes

$$\begin{aligned} q &= \frac{(1+p_l^2)q_l \pm R\sqrt{(1+p_l^2)(1+p_l^2+q_l^2-R^2)}}{R^2 - q_l^2}, \\ p &= p_l. \end{aligned} \tag{12}$$

b) When $\phi = \frac{2N+1}{2}\pi$ (N : integer)

In the same way, $q = q_l$ because $\cos\phi = 0$ and $\sin\phi = \pm 1$. The solution becomes

$$\begin{aligned} p &= \frac{(1+q_l^2)p_l \pm R\sqrt{(1+q_l^2)(1+p_l^2+q_l^2-R^2)}}{R^2 - p_l^2}, \\ q &= q_l. \end{aligned} \tag{13}$$

c) When $R_o = 1$, that is at classical singular point

Because in eq. (11) A,B and D become, respectively,

$$A \equiv \sec^2\phi + (p_l + q_l\tan\phi)^2,$$

$$B \equiv q_l\left(\sec^2\phi + (p_l + q_l\tan\phi)^2\right),$$

$$D \equiv 0,$$

the non-singular solution which is consistent with the intuitive result becomes

$$p = p_l, \qquad q = q_l. \tag{14}$$

d) When $R_o = 0$, that is in occluding boundary

When $\phi \neq \frac{N}{2}\pi$ (N : odd integer), using $(1 + p_l p + q_l q) = 0$ and eq. (9)

$$\begin{aligned} q &= -\frac{1 + p_l(p_l + q_l\tan\phi)}{q_l - p_l\tan\phi}, \\ p &= -q\tan\phi + (p_l + q_l\tan\phi). \end{aligned} \tag{15}$$

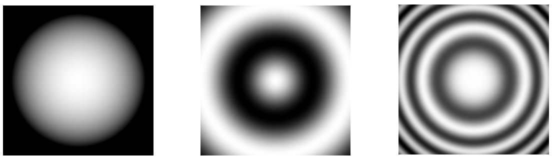
When $\phi = \frac{N}{2}\pi$ (N : odd integer), using the fact that $q = q_l$ and eq. (9)

$$q = p_l, \qquad p = -\frac{1+q_l^2}{p_l}. \tag{16}$$

e) When ϕ can't be defined

This corresponds to the special case that the surface is locally a plane, that is both the principal curvatures κ_1 and κ_2 are identically zero. The solution of this point should be interpolated from other points having the analytic solution because the solution is not exist by our approach.

The derived solutions exist at every point of image when the surface is not a plane locally. That



(a) (b) (c)

Figure 7: Depth images used for simulations.

$\vec{L}$	$\triangle p$	$\triangle q$
L_1	0.59	0.59
L_2	1.18	1.18
L_3	1.18	1.18

(a)

$\vec{L}$	$\triangle p$	$\triangle q$
L_1	0.73	0.73
L_2	0.74	0.74
L_3	0.73	0.73

(b)

$\vec{L}$	$\triangle p$	$\triangle q$
L_1	0.04	0.04
L_2	0.00	0.00
L_3	0.01	0.00

(c)

L_1=(0.0,0.0,3.0), L_2=(1.0,1.0,3.0), L_3=(−0.7,0.0,3.0

Figure 8: Error (%) calculated by analytical solution.

is, these are analytic solutions, and neither iteration nor global propagation starting from a special condition is needed. Therefore, we believe that the proposed scheme is superior to conventional approaches. The uniqueness of solution is guaranteed only at classical singular points, which are not singular in actual, and in occluding boundary; therefore there generally exist two independent solutions which are ambiguous due to the non-linear property of the SFS problem. The solutions are weak because these are basically C^1 continuous with respect to C^2 continuous surface; therefore the solutions may cause topological problem in the integration process[19] at critical points when the role of convex/concave with respect to the position of light source is switched.

4 Simulations

Simulations have been done for three depth images, as is shown in Fig. 7, having approximately C^2 continuous surface. It is assumed, in the reconstruction process, that the angle ϕ can be obtained by another algorithm because the purpose of this paper is in showing the local existence of analytical solution of the SFS problem. The angle is calculated by Eq. (4) using depth images.

Three cases of light conditions have been imposed and reconstruction errors have been calculated as is shown in Fig. 8. Simulated errors have been calculated for slopes p and q, respectively, by the equation

$$\Delta p(\%) = 100. \times |\frac{p_{real} - p_{calc}}{p_{real}}|. \qquad (17)$$

The simulation results show the correctness of derived analytic solution.

5 Discussions and conclusions

We discussed the concept of a new geometrical approach in solving the SFS problem and derived the C^1 continuous weak analytic solutions of the problem.

Several ad hoc constraints have been used in numerous approaches by assuming that the SFS problem is ill-constrained. However, we showed that i) this problem is bound by two constraints, two angles θ and ϕ which come from the definition of the problem itself, and ii) two constraints are orthogonal each other.

Consequently, our work indicates that ad hoc constraints is needless. Actually, the local existence of analytical solution is trivial when the problem is analyzed by elementary geometry as is shown in Fig. 6. The contribution of this paper is based on well known mathematical works[4, 16] except the fact that the SFS problem has local analytical solutions contrary to general cases of non-linear PDE of first order.

It is not difficult to show that ad hoc constraints are needless. For example, the constraint of occluding boundary[15] is just a natural result of the analytic solution as has been pointed out by a previous research[23]. It is already known that several constraints, classified as a sort of smoothness constraints, cause over-smoothed results in some cases. This corresponds to a typical mathematical fault which solves a problem by using non-orthogonal constraints because the smoothness or the continuity already constrains the problem by assuming the existence of local slopes, i.e. that of the problem itself, which are defined on smoothly varying domain. Whereas, we resolved the assumption of the problem and showed that the problem is composed of two orthogonal constraints.

We showed that a set of C^1 continuous analytic solutions exists at any point of observed image even at classical singular points and in occluding boundaries. This proves the existence of solution. The solution is weak because the C^1 continuous solution is obtained with respect to the C^2 continuous surface. The weakness may cause a topological problem in the integration process[19]; i.e. we should select one prefer solution, using the condition of C^2 continuity and using another assumption about convexity or concavity of object, between two possible combinations: (p, q_+) and (p, q_-). This indicate that the SFS problem is not unique but has only two degrees of freedom due to the non-linear property of the problem.

We believe that the derivation of this analytic solution has significant meaning in the fact that this solution makes it possible to calculate the tangent surface of object using shade information fastly and reliably without doing iterations when constant albedo of surface is already known. The way of determining one constraint ϕ will be discussed in another paper because it needs a precise and comprehensive approach.

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