Proceedings of the 4th IEEE Conference on Control Applications, Albany, USA, Sep. 1995, pp. 500-505.

# DYNAMIC COUPLING OF UNDERACTUATED MANIPULATORS

**Marcel Bergerman** **Christopher Lee** **Yangsheng Xu**

The Robotics Institute
Carnegie Mellon University
Pittsburgh PA 15213
{mbergerm|chrislee|xu+}@cs.cmu.edu

### Abstract

In recent years, researchers have been dedicated to the study of underactuated manipulators which have more joints than control actuators. In previous works, assumptions were made as to the existence of enough dynamic coupling between the active and the passive joints of the manipulator for it to be possible to control the position of the passive joints via the dynamic coupling. In this work, the authors aim to develop an index to measure the dynamic coupling, so as to address when control of the underactuated system is possible, and how the motion and robot configuration can be designed. We discuss extensively the nature of the dynamic coupling and of the proposed coupling index, and their applications in the analysis and design of underactuated systems, and in control and planning of robot motion configuration.

## 1 Introduction

In recent years, researchers have been turning their attention to so called underactuated systems, where the term *underactuated* refers to the fact that the system has more joints than control actuators. Some examples of underactuated systems are hyper-redundant (snake-like) robots with passive joints; legged robots with passive joints; robot manipulators with failed actuators; free-floating space robots, where the base can be considered as a virtual passive linkage in inertia space; etc. The importance of the study of underactuated systems is demonstrated by the following examples: manipulators with passive joints and hyper-redundant robots with few actuators are important from the viewpoint of energy saving, lightweight design and compactness; for regular manipulators, a joint failure (and consequent underactuation) can compromise an entire operation; finally, in free-floating space systems, the base (satellite) can be considered as a 6-DOF device without positioning actuators whose attitude must be controlled.

Recent works presented control strategies to deal with the joint and Cartesian control problem of underactuated manipulators. Arai and Tachi [1] presented a method which required brakes to be installed on each passive joint. The basic idea was to use the dynamic coupling between the active and the passive joints to drive the passive joints to a desired set-point. Later, Bergerman and Xu [3] enhanced this method to deal with parameter uncertainty and provide the system with a greater deal of robustness. This is specially important in these systems because the Jacobian mapping from Cartesian to joint space depends on the dynamic parameters, as opposed to conventional robots, where the Jacobian depends solely on kinematic parameters. Parameter uncertainty can lead the system to poor or even unstable performance. Another control strategy requiring the use of brakes was developed by Papadopoulos and Dubowsky [11] for a space manipulator.

In every work mentioned above, the authors had to assume that sufficient coupling existed between the passive and the active joints, so that the controller could “transmit” the forces/torques to the passive joints in order to drive them. However, no attempts were made to quantify this coupling, or to identify the cases where it is too small as to be practically unfeasible to control the passive joints. To give the reader an introductory feeling of the importance of the dynamic coupling in underactuated mechanisms, consider a Cartesian 3-DOF manipulator, where the joint axes are mutually perpendicular. It can be verified that the inertia matrix for this manipulator is diagonal and constant, corresponding to the physical fact that there is no coupling at all between the joints. No matter how much one joint travels, the other ones are unaffected. Consequently, the absence of coupling does not allow this mechanism to be controlled at all if passive joints are present.

In this work, the authors aim to provide a measure of the dynamic coupling present in underactuated systems. This measure is useful not only for the design of an underactuated manipulator, so as to maximize the coupling and hopefully minimize the energy necessary to perform control; it is also useful for such important issues as actuator placement and control strategies. In cases for which the number of actuators is greater than the number of passive joints, such a measure can also be used in connection with a redundant control scheme [9], in order to maintain the system as far as possible from the positions that yield low dynamic coupling.

## 2 Dynamic Coupling

In order to derive the measure of dynamic coupling between the accelerations of the passive and active joints of an underactuated manipulator, we must first present the dynamic equations governing the behavior of the system. Let $n$ be the total number of joints, $r$ the number of active joints, and $p = n - r$ the number of passive ones. By using the Lagrangian formulation [4], one can obtain the following set of differential equations relating the accelerations of the joints to the torques supplied by the actuators:

$$M_0(q_0)\ddot{q}_0 + b_0\left(q_0, \dot{q}_0\right) = \tau_0 \tag{1}$$

Here, the matrix $M_0$ is the $n$ x $n$ inertia matrix of the manipulator, $b_0$ is a vector containing all the centrifugal, Coriolis and gravitational torques, and $\tau_0$ is the vector of torques applied at the active joints. Note that $\tau_0$ has always $p$ components equal to zero, corresponding to the absence of actuators at the passive joints.

Since Equation (1) does not reveal the relationship between the active and the passive joints' accelerations, we partition it as:

$$\begin{array}{cc} & \begin{array}{cc} r & p \end{array} \\ \begin{array}{c} r \\ p \end{array} & \begin{bmatrix} M_{aa} & M_{ap} \\ M_{pa} & M_{pp} \end{bmatrix} \end{array} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_p \end{bmatrix} + \begin{bmatrix} b_a \\ b_p \end{bmatrix} = \begin{bmatrix} \tau_a \\ 0 \end{bmatrix} \tag{2}$$

where $q_a$ correspond to the active joint angles, $q_p$ to the passive ones, and $q = [q_a^T \ \ q_p^T]^T$. It must be noted that $M$ as defined in (2) *is not always equal* to the conventional inertia matrix of mechanical manipulators. Nonetheless, $M$ still preserves important properties of the original inertia matrix, such as symmetry and positive-definiteness [2].

The submatrix $M_{pp}$, being a square diagonally-positioned submatrix of a symmetric positive-definite matrix, can be proved to be positive-definite. From the second line of (2), we can then write:

$$\ddot{q}_p = -M_{pp}^{-1}M_{pa}\ddot{q}_a - M_{pp}^{-1}b_p \tag{3}$$

The second term on the right-hand side of (3) is a function only of $q$ and $\dot{q}$, and as such is completely determined once measurements of these variables are available. Because of the impossibility of integrating, in the general case, the nonholonomic constraints imposed by the passive joints (as demonstrated by Oriolo and Nakamura [10]), we focus on the acceleration relationship between the active and the passive joints, rewriting equation (3) as:

$$\ddot{\bar{q}}_p = -M_{pp}^{-1}M_{pa}\ddot{q}_a \equiv M_c\ddot{q}_a \tag{4}$$

where

$$\ddot{\bar{q}}_p = \ddot{q}_p + M_{pp}^{-1}b_p \tag{5}$$

The acceleration $\ddot{\bar{q}}_p$ can be viewed as a virtual acceleration of the passive joints, generated by the acceleration of the active ones, and by the nonlinear torques due to velocity effects.

Equation (4) is important in the understanding of how an underactuated system works. Torques can only be applied at those joints which contain an actuator, or the active joints. These torques produce the accelerations $\ddot{q}_a$, which indirectly produce the accelerations $\ddot{q}_p$ at the passive joints. The passive joints' accelerations can only be controlled if the $p$ x $r$ matrix $M_c$ possesses a structure that allows the torques to be transmitted reasonably "well" (in a sense to be defined later) to the passive joints. Thus, the study of this matrix is of fundamental importance for the design and control of underactuated manipulators.

To begin the analysis, note that matrix $M_c$ is a function only of the robot's configuration $q$, and thus is completely determined based on the readings of the encoders in all joints. It does not depend on $\ddot{q}_a$ or $\ddot{q}_p$. Thus, equation (4) can be regarded as a linear system of $p$ equations $Ax = b$, underconstrained for $r > p$ and overconstrained for $r < p$.

## 3 Dynamic Coupling Measure

We can study the relationship between the accelerations of the passive and active joints using Equation (4). According to our definition, $M_c$ is a $p$ x $r$ matrix. We must study the various possibilities that can arise depending on whether there are more active or more passive joints in the mechanism. The first consideration that can be made regards the rank of matrix $M_c$. It is known that this rank obeys:

$$\text{rank}\,(M_c) \le \min\,(p, r) \tag{6}$$

This fact will be used in the sequence.

- Case 1: $r < p$

When the number of actuators is smaller than the number of passive joints, $M_c$ has maximum rank $r$ and equation (4) has *at most* one solution. However, this solution (if it exists) is not interesting in practice, because the accelerations of $p - r$ passive joints will depend linearly on the accelerations of the other $r$ passive joints. In other words, $r$ passive joints can be controlled at every instant, while the other $p - r$ of them cannot. We can conclude that it is necessary to have *at least* $p$ actuators in the underactuated mechanism to be possible to control all $p$ passive joints independently. Note that this result was already established by Arai and Tachi [1]; however in their work, this conclusion was reached only after a study of the linearized dynamic equations of the system.

• Case 2: $r = p$

In this case we can obtain at most one solution, which exists if the $r$ x $r$ matrix $M_c$ is invertible. A case-by-case pre-analysis of $M_c$ can show whether it will be possible to control $\ddot{q}_p$ using the actuators at the active joints.

• Case 3: $r > p$

This is probably the most common case, and certainly the most interesting one. Here, we can obtain *at least* one solution for the problem of finding the $\ddot{q}_a$ that will generate the desired $\ddot{q}_p$, provided the rank of matrix $M_c$ is at least equal to $p$. In the general case, infinite solutions can be found. One can choose among these solutions the one that provides the minimum norm of $\ddot{q}_a$, so as to save energy, or effectively make use of this redundancy to accomplish tasks such as obstacle avoidance, coupling maximization, actuability maximization [7], etc.

In any of the cases above, it is useful to define a measure of the dynamic coupling at any given instant. Following Lee and Xu [7] and Xu and Shum [12], it is natural to think of the singular values of $M_c$, which quantify its "degree of invertibility" and thus its capacity to "transmit" the torque from the active to the passive joints. Based on this rationale, let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_l$ be the $l = \min(p, r)$ singular values of $M_c$. We define the measure of dynamic coupling as:

$$\rho_c = \prod_{i=1}^{l} \sigma_i \tag{7}$$

We call $\rho_c$ as above the *coupling index* of the underactuated manipulator. As will be shown in the sequence, the coupling index can be used as a design tool for actuator placement, desirable robot configuration, or as a quantity to be used during the real-time control of the manipulator.

**Example 1** Consider the 2-link planar manipulator shown in Figure 1. Considering joint 1 active, and joint 2 passive, we have:

$$M_c = -\left[1 + \frac{m_2 l_1 l_{c_2} \cos(\theta_2)}{m_2 l_{c_2}^2 + I_2}\right] \tag{8}$$

where the symbols $m$, $l$, $l_c$, and $I$, correspond to the links' mass, length, location of the center of mass and inertia, respectively. Adopting the following dynamic parameters: $m_1$ = 2Kg; $I_1$ = 0.2 Kg m$^2$; $m_2$ = 1Kg; $I_2$ = 0.1 Kg m$^2$; $l_1 = l_2$ = 0.3m; $l_{c1} = l_{c2}$ = 0.15m; we obtain:

$$\rho_c = |1 + 0.37\cos\theta_2| \tag{9}$$

We see that, for this manipulator, the matrix $M_c$ is *always* invertible, and thus control of the passive joint via the dynamic coupling is always possible. Based on the present study, one can now pre-analyze the system in order to determine whether or not control is possible, before making any attempt to control it. ■

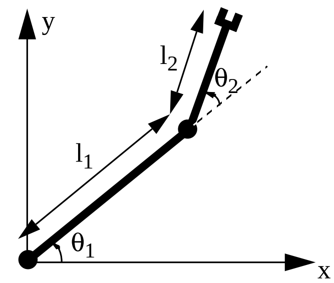


**Figure 1:** Two-link planar manipulator.

## 4 Implementation Issues

### 4.1 Manipulator Design

The coupling index derived previously can be used effectively on the design of the underactuated system as a mathematical tool that determines the optimal actuator placement of an underactuated manipulator, as the next example shows.

**Example 2** Consider a 3-DOF planar manipulator with rotary joints, as shown in Figure 2. We adopt the same parameters as in example 1, and adopt for link 3 the same mass, length and inertia of link 2. We assume also $r = 2$, i.e., we have two actuators to be placed either on joints 1 and 2 (case 1), 1 and 3 (case2) or 2 and 3 (case 3).

Figure 3 shows the value of $\rho_{c_i}, i = 1, 2, 3$ as a function of $\theta_2$ and $\theta_3$ (the indexes 1, 2 and 3 differentiate each case). A careful consideration of this figure shows that, for most values of the joint angles, $\rho_{c_1}$ is the greatest index of all three. This can be verified by the values in Table 1. As we see, in none of the cases does $\rho_{c_i}$ becomes zero (or "dangerously" close to zero). This indicates that at least one solution will always exist, no matter which joint is the passive one. Also, the choice of joint 3 as the passive joint increases the dynamic coupling, and enhances the control of the passive joint by the active ones. ■

**Table 1:** Maximum, minimum, average and standard deviation values attained by $\rho_{c_i}, i = 1, 2, 3$.

| $i$ | $\max(\rho_{c_i})$ | $\min(\rho_{c_i})$ | $\mathrm{avg}(\rho_{c_i})$ | $\mathrm{std}(\rho_{c_i})$ |
|---|---|---|---|---|
| 1 | 2.0021 | 0.8576 | 1.4211 | 0.3033 |
| 2 | 1.4774 | 0.6645 | 1.0607 | 0.2811 |
| 3 | 0.5104 | 0.3912 | 0.4556 | 0.0374 |

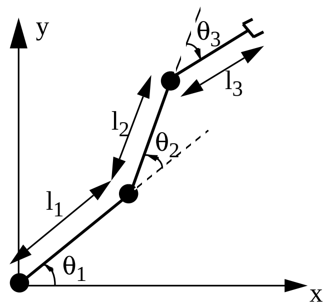


**Figure 2:** Three-link planar manipulator.

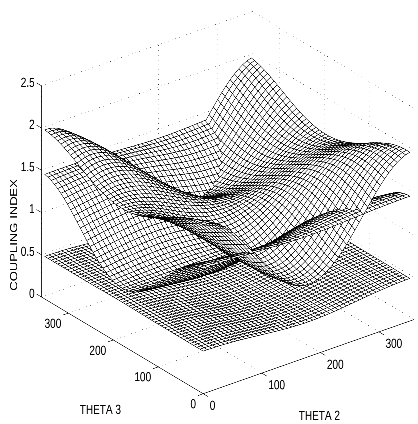


**Figure 3:** Coupling indices for the 3-link manipulator.

Note how this approach differs from the one studied by Lee and Xu [7], where the authors defined the *actuability index* of underactuated manipulators. The actuability index measures the arbitrariness of the actuator's ability to cause acceleration at the end-effector. Thus, *it relates torques in the active joints and accelerations at the end-effector in Cartesian space*, while the coupling index defined in this work *relates accelerations of the active joints to accelerations of the passive ones*. The conclusions derived in that work and the ones here should not be compared, for the indexes operate in different manners. To be more specific, the coupling index indicates how much acceleration is possible to be obtained at the passive joints given limited accelerations at the active joints. There is no attempt to quantify the accelerations possible to be obtained at the end-effector. The actuability index indicates how much acceleration can be obtained at the end-effector given limited torques at the active joints. It does not attempt to quantify the accelerations at the passive joints. We refer the reader to Lee and Xu's work [7] for more detailed information on the actuability index.

### 4.2 Individual Joint Coupling

In the same way we defined the coupling index, one can think of an *individual joint coupling index* relating each passive joint to each active one. Working with these pairs of active-passive joints, we can measure the dynamic coupling existent between each of them. Ultimately, our objective will be that of determining which active joint should be chosen to drive each passive one.

Recall equation (4):

$$\ddot{q}_p = M_c \ddot{q}_a \tag{10}$$

If we split the expression above on its constituting lines, we have:

$$\ddot{q}_{p_i} = M_{c_{i,1}} \ddot{q}_{a_1} + \ldots + M_{c_{i,r}} \ddot{q}_{a_r} \tag{11}$$

In order to study the coupling between the *i*-th passive and the *j*-th active joint, we can assign zero values to all except the *j*-th active joint's acceleration:

$$\ddot{q}_{p_i} = M_{c_{i,j}} \ddot{q}_{a_j} \tag{12}$$

This equation resembles (4) and so we can define the *coupling index between the i-th passive and the j-th active joint*:

$$\rho_{c_{ij}} = \left| M_{c_{i,j}} \right| \tag{13}$$

For every row in the matrix $M_c$, the element $(i, j)$ with greater magnitude will indicate that the greatest dynamic coupling for the passive joint *i* comes from the active joint *j*.

**Example 3** Consider the system shown in Figure 2, where joints 1 and 2 are active and joint 3 is passive (according to the results in example 2). For this system, the matrix $M_c$ is given by:

$$M_c = -\begin{bmatrix} 1 + 0.2727\,(c_{23} + c_3) & 1 + 0.2727 c_3 \end{bmatrix} \tag{14}$$

According to (13) we have:

$$\begin{aligned} \rho_{c_{11}} &= \left| 1 + 0.2727\,(c_{23} + c_3) \right| \\ \rho_{c_{12}} &= \left| 1 + 0.2727 c_3 \right| \end{aligned} \tag{15}$$

Table 2 presents the maximum, minimum, average and standard deviation values for both $\rho_{c_{11}}$ and $\rho_{c_{12}}$. For this system it is difficult to tell which active joint has greater coupling with the passive one. Although $\rho_{c_{11}}$ attains greater values than $\rho_{c_{12}}$, it also attains smaller values; and the averages of both indexes are the same. This example shows that the coupling index, although useful, may not provide the control engineer with sufficient information to decide on which active joint should drive the passive one. Global indexes, based on the integral of the coupling index, may be more useful in this case, as it will be shown in the sequence. ■

**Table 2:** Maximum, minimum, average and standard deviation values attained by $\rho_{c_{11}}$ and $\rho_{c_{12}}$.

| $i$ | $\max(\rho_{c_i})$ | $\min(\rho_{c_i})$ | $\mathrm{avg}(\rho_{c_i})$ | $\mathrm{std}(\rho_{c_i})$ |
|---|---|---|---|---|
| 11 | 1.5454 | 0.4546 | 1.0000 | 0.2728 |
| 12 | 1.2727 | 0.7273 | 1.0000 | 0.1929 |

### 4.3 Global Coupling Index

The coupling index defined previously is a local measure; it measures the "amount" of dynamic coupling between the active and the passive joints at every point in the joint space. The greater the coupling index at a particular configuration of the manipulator, the easier for the active joints to drive the passive ones. However, in design and path planning problems, one is more interested in the coupling index in a global sense, so as to measure the dynamic coupling existent within the workspace of the manipulator. In this case, a *global coupling index* is more useful, for it will measure the coupling between the joints at *all* points in the joint space. One possible way of defining a global coupling index is as follows [5]:

$$\rho_c^g = \frac{\int_\Theta \rho_c^2 d\Theta}{\int_\Theta d\Theta} \tag{16}$$

where the integrals above are taken over the entire joint space $\Theta \in \Re^n$ of the manipulator. The use of the squared value of the coupling index is inert because the coupling index is always non-negative; and this choice facilitates the problem of finding a closed-form solution to the above integral, because the singular values of a matrix $A$ are equal to the positive square roots of the nonzero eigenvalues of $A^T A$.

This choice of the global coupling index will take into account not only the local coupling between the joints, but that available over the entire joint space, as the next examples show.

**Example 4** We re-analyze now example 2, where the objective was to determine the best actuator placement based on the dynamic coupling between the joints. There, the decision to place the actuators on joints 1 and 2 was based on the maximum, minimum and average values of $\rho_c$ for the various possible configurations. Here we can base this decision on a global index, which already takes into account all these quantities.

We considered in the following calculations that the joints can rotate freely around their respective axes from 0 to $2\pi$. In cases where there are physical joint limits, this can be taken into account in the calculation of the integrals in (16). For cases 1, 2, and 3 studied in example 2, we have the results shown in table 3. We can immediately conclude that case 1 is the one which provides greater dynamic coupling between the active and the passive joints in a global sense. Note that this is the same conclusion as the one drawn in example 2, reached in a much simpler way. ■

**Table 3:** Global coupling index $\rho_{c_i}^g$, $i = 1, 2, 3$.

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $\rho_{c_i}^g$ | 2.1115 | 1.2041 | 0.2090 |

**Example 5** The global coupling index can also be used in conjunction with the individual joint coupling index defined in section 4.2. The idea is to use the later instead of $\rho_c$ in equation (16):

$$\rho_{c_{ij}}^g = \frac{\int_\Theta \rho_{c_{ij}}^2 d\Theta}{\int_\Theta d\Theta} \tag{17}$$

Calculating the above *global individual joint coupling index* for the manipulator in example 3, we obtain:

$$\rho_{c_{11}}^g = \rho_{c_{12}}^g = 1.0372 \tag{18}$$

It can be seen that the global coupling between the first and second active joints and the passive joint is exactly the same. For control purposes, then, one can choose either active joint to dynamically control the passive joint. Note again that this conclusion was drawn immediately, as opposed to example 3, where no conclusion could be drawn. ■

**Example 6** The coupling index is also useful for the purpose of designing the links of an underactuated mechanism. Consider the 2-dof manipulator in Figure 1, with joint 1 active, and parameters as in example 1. We want to determine the center of mass of link 2, $l_{c_2}$, so as to maximize the global coupling index. In this case we have:

$$\rho_c^g = \frac{1 + 70 l_{c_2}^2 + 100 l_{c_2}^4}{1 + 20 l_{c_2}^2 + 100 l_{c_2}^4} \tag{19}$$

Figure 4 shows the global coupling index as a function of $l_{c_2}$. As we can see, it attains the maximum value $\rho_c^g = 2.250$ for $l_{c_2} = 0.316\mathrm{m}$. This example shows how the global coupling index can be used for design issues other than actuator placement. ■

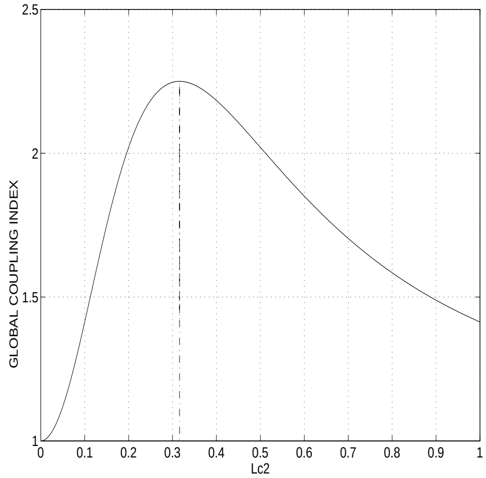


**Figure 4:** Global coupling index in example 6.

## 5 Conclusion

There has been considerable progress recently in the area of analysis and control of underactuated manipulators. The current literature in this area however, has always assumed that sufficient dynamic coupling was available between the active and passive joints for effective control of the manipulator. Because this assumption is not always valid, it is vital for design, analysis, and control of these systems that the amount of available coupling be quantified.

In this work, the authors have proposed a measure of the dynamic coupling between the active and passive joints of an underactuated manipulator. The coupling index indicates precisely the amount of coupling available for the purpose of controlling the passive joints of the manipulator through the application of torques in the active joints. We have also proposed a global coupling index, which indicates the amount of coupling available over the entire workspace of the manipulator.

As we have shown through a series of illustrative examples, the proposed indices can be used effectively within the contexts of mechanical design, configuration optimization, and real-time control. Moreover, when redundant degrees of freedom are available in an underactuated manipulator, these indices may be used as local measures for use within a redundant control scheme.

## 6 Acknowledgments

We would like to acknowledge the discussions with Christiaan Paredis, Carnegie Mellon University. The first author is supported by a grant from the Brazilian National Council for Research and Development (CNPq).

## 7 References

[1] H. Arai and S. Tachi, "Position control of a manipulator with passive joints using dynamic coupling", *IEEE Transactions on Robotics and Automation*, vol. 7, no. 4, Aug. 1991, pp. 528-534.

[2] M. Bergerman, C. Lee and Y. Xu, "Dynamic coupling of underactuated manipulators", *CMU-RI-TR-94-25*, Carnegie Mellon University, Aug. 1994.

[3] M. Bergerman and Y. Xu, "Robust control of underactuated manipulators: analysis and implementation", *Proc. of the IEEE Systems, Man and Cybernetics Conference*, Oct. 1994.

[4] J. J. Craig, *Introduction to robotics*: Mechanics and control, Addison-Wesley, Reading, 2 ed., 1989.

[5] C. Gosselin and J. Angeles, "A global performance index for the kinematic optimization of robotic manipulators", *Transactions of the ASME, Journal of Mechanical Design*, vol. 113, Sep. 1991, pp. 220-226.

[6] P. Lancaster and M. Tismenetsky, *The theory of matrices*, Academic Press, Orlando, 2 ed., 1985, pp. 152.

[7] C. Lee and Y. Xu, *Actuability of underactuated manipulators*, CMU-TR-RI-94-13, Carnegie Mellon Univ., May 1994.

[8] R. Mukherjee and D. Chen, "Control of free-flying underactuated space manipulators to equilibrium manifolds", *IEEE Trans. on Robotics and Automation*, vol. 9, no. 5, Oct. 1993.

[9] Y. Nakamura, *Advanced robotics*: Redundancy and optimization, Addison-Wesley, Reading, 1ed., 1991.

[10] G. Oriolo and Y. Nakamura, "Control of mechanical manipulators with second-order nonholonomic constraints: underactuated manipulators", *Proc. of the 30th Conference on Decision and Control*, Dec. 1991, pp. 2398-2403.

[11] E. Papadopoulos and S. Dubowsky, "Failure recovery control for space robotic systems", *Proceedings of the 1991 American Control Conference*, vol. 2, 1991, pp. 1485-1490.

[12] Y. Xu and H. Y. Shum, "Dynamic control and coupling of free-floating space robot systems", to appear in *Journal of Robotic Systems*.

[13] T. Yoshikawa, *Foundations of robotics*: Analysis and control, MIT Press, Cambridge, 1ed., 1990.