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MECHANICS AND CONTROL OF A TERRESTRIAL VEHICLE EXPLOITING A NONHOLONOMIC CONSTRAINT FOR FISHLIKE LOCOMOTION

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ABSTRACT

We present a novel mechanical system, the "landfish," which takes advantage of a combination of articulation and a nonholonomic constraint to exhibit fishlike locomotion. We apply geometric mechanics techniques to establish the equations of motion in terms of the system's nonholonomic momentum and analyze the system's equilibrium properties. Finally, we demonstrate its locomotion capabilities under several controllers, including heading and joint velocity control.

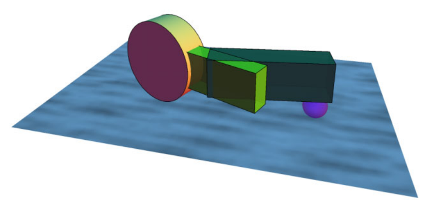

Figure 1. THE "LANDFISH," CONSISTING OF TWO LINKS, A NONHOLONOMICALLY CONSTRAINED WHEEL, AND CASTER.

1 INTRODUCTION

Nonholonomic vehicles have widely been studied for their ability to locomote solely via interactions with their environment. Many canonical examples are extensions of the Chaplygin sleigh [1], a rigid body supported by a wheel and casters, with the constraint that the wheel cannot slip laterally. This paper introduces the "landfish," shown in Fig. 1. Like the sleigh, the body has a nonholonomically constrained wheel at the rear, but this system consists of two connected rigid bodies, with the front link being able to rotate freely about the back.

The landfish generalizes several Chaplygin sleigh variations in the literature. Osborne and Zenkov [2] studied a sleigh with a moving point mass around the sleigh's center of mass, but the present system decouples the mass distribution imbalance from the rotational inertia imbalance. Fairchild et al. [3] demonstrated proportional heading control for the previous system in which the mass was constrained to move laterally. Kelly et al. [4] extended proportional control to the Chaplygin beanie, in which a body with rotational inertia sat on top of the sleigh; in contrast, the landfish's front link is offset from the rear's center of mass.

In addition to introducing new complexity to these previously studied systems, this paper clarifies the link between terrestrial locomotion exploiting a rolling constraint and the model for fishlike swimming introduced by Kelly and Xiong [5]. In each case, a body with a single internal degree of freedom associated with lateral flexing is able to propel itself because a velocity constraint—in the case of the swimming hydrofoil, a Kutta condition—is enforced at one end of the body but not at the other. The constraint breaks the symmetry that would result in the conservation of linear and angular momentum were the body unconstrained, allowing cyclic variations in the body's shape to be rectified as forward motion.

The problem of steering these Chaplygin systems was studied by [3, 4]. Both systems have a single controlled degree of

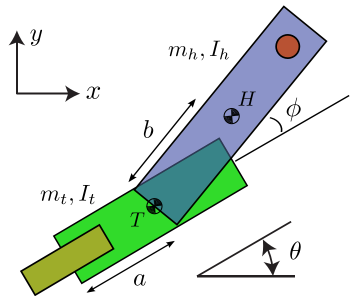

Figure 2. THE CONFIGURATION OF THE LANDFISH.

freedom, the manipulation of which can regulate the systems' heading and generate forward locomotion. The control objective was to achieve steady translation in a specified direction. In both cases, single input proportional feedback control was shown to be effective in stabilizing the systems toward a specified heading, a simple result despite the nonlinearities of such systems.

2 THE LANDFISH MODEL

Figure 2 depicts the landfish schematically with its parameters. Denote the center of mass of the *tail* link T, and that of the *head* link H. The end of the head link is connected via a hinge to the tail link at T. The distance from the nonholonomic back wheel's ground contact point to T is a, while the distance from T to H is b. The head is supported by an unconstrained caster.

The symbols m_t, I_t, m_h, and I_h represent the masses of the tail and head links, respectively, and their moments of inertia relative to vertical axes passing through their centers of mass. We also define the total mass $M = m_h + m_t$ and total inertia $I = I_h + I_t$ for convenience. The wheel and caster are both assumed to be massless, and the system is assumed to remain upright at all times. The two links are depicted as rectangular prisms that are free to pass through one another.

The system's configuration is given by $q = (g, r) \in Q$, where the position variables $g = (x, y, \theta) \in G = SE(2)$ locate the landfish in the world and the shape variable $r = \phi \in M = \mathbb{S}^1$ describes the head link's orientation relative to the tail. The variables x and y denote the global position of the tail link's center of mass, while θ denotes the tail link's orientation with respect to the horizontal. In the context of geometric mechanics, $Q = M \times G$ is a *trivial principal fiber bundle*, composed of the *base space* M and the *fiber space* G.

3 LANDFISH MECHANICS

We analyze the landfish as a constrained Lagrangian system, invoking nonholonomic reduction [6] to reduce the standard diffential equations of motion to a pair of equations governing the evolution of the *nonholonomic momenta*. The Lagrangian itself is invariant under the tangent lift of the action corresponding to left translation in $SE(2)$. The Lagrangian is

$$L = \frac{m_t(\dot{x}^2 + \dot{y}^2) + m_h(\dot{x}_h^2 + \dot{y}_h^2) + I_t\dot{\theta}^2 + I_h(\dot{\theta} + \dot{\phi})^2}{2}, \qquad (1)$$

where

$$\dot{x}_h = \dot{x} - b\sin(\theta + \phi)(\dot{\theta} + \dot{\phi}), \quad \dot{y}_h = \dot{y} + b\cos(\theta + \phi)(\dot{\theta} + \dot{\phi}).$$

The no slip condition on the back wheel gives rise to the system's nonholonomic constraint. Similar to the Lagrangian, it is invariant under the cotangent lifted action. It can be written in *Pfaffian form* $\omega = \omega(q)$, where

$$\omega(q) = -\sin\theta\, dx + \cos\theta\, dy - a\, d\theta \qquad (2)$$

is the constraint one-form that lives in the cotangent space of the configuration manifold Q. The constraint requirement is that the system's generalized velocity $\dot{q}$, which is tangent to Q, annihilates ω, such that $\omega(q)\dot{q} = 0$.

From Eq. (1) and Eq. (2), one may compute the *nonholonomic momentum*, which projects the *generalized momentum* into a basis that respects the nonholonomic constraint. Such a projection can be done in more than one way; we choose a pair of vector fields on Q to span the space

$$S_q = D_q \cap T_q\mathrm{Orb}(q).$$

Here, D_q comprises all tangent vectors annihilated by ω (effectively its null space), while $T_q\mathrm{Orb}(q)$ denotes the space tangent to the orbit of the group action. We thus choose

$$S_q = \mathrm{span}\left\{\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, -a\sin\theta\frac{\partial}{\partial x} + a\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}\right\} \qquad (3)$$

Flow along the first vector field corresponds to translation parallel to the rear wheel, while flow along the second corresponds to rotation about the wheel's ground contact point.

Each of the vector fields in Eq. (3) constitutes the infinitesimal generator ξ_Q of the $SE(2)$ action corresponding to a particular element ξ of the Lie algebra $\mathfrak{se}(2)$. By definition, the associated component J of the nonholonomic momentum is

$$J = \frac{\partial L}{\partial \dot{q}^i}(\xi_Q)^i \qquad (4)$$

where q^i is the ith coordinate on Q. Evaluating Eq. (4) for each of the components of S_q, we obtain two different momenta in linear and angular components:

$$\begin{aligned}
J_{LT} =& M(\dot{x}\cos\theta + \dot{y}\sin\theta) - bm_h \sin\phi(\dot{\theta}+\dot{\phi}),\\
J_{RW} =& -(aM\sin\theta + bm_h\sin(\theta+\phi))\dot{x}\\
&+(aM\cos\theta + bm_h\cos(\theta+\phi))\dot{y}\\
&+I\dot{\theta} + I_h\dot{\phi} + bm_h(b+a\cos\phi)(\dot{\theta}+\dot{\phi}).
\end{aligned} \tag{5}$$

The evolution of the momenta can be found from the following differential equation:

$$\dot{J} = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d\xi}{dt}\right]^i_Q \tag{6}$$

Thus, if the evolution of the joint angle $\phi(t)$ is specified, then the dynamics of the system are determined completely by a system of equations of the form

$$\begin{aligned}
\dot{J}_{LT} &= f(J_{LT}, J_{RW}, \phi, \dot{\phi}),\\
\dot{J}_{RW} &= g(J_{LT}, J_{RW}, \phi, \dot{\phi}).
\end{aligned} \tag{7}$$

The evolution of $x(t)$, $y(t)$, and $\theta(t)$ can then be reconstructed from the solution to these equations.

The nonholonomic momentum equations (7) may be written relatively compactly if we allow ourselves to include the symbol $\dot{\theta}$ as an additional argument to the functions f and g. In this case, the equations of motion for the system may be written as

$$\begin{aligned}
\dot{J}_{LT} &= \dot{\theta}\left(aM\dot{\theta} + bm_h\cos\phi(\dot{\theta}+\alpha)\right)\\
\dot{J}_{RW} &= -\frac{1}{M}\left(aM\dot{\theta} + bm_h\cos\phi(\dot{\theta}+\alpha)\right)\\
&\qquad\qquad \left(J_{LT} + bm_h\sin\phi(\dot{\theta}+\alpha)\right)\\
\dot{\theta} &= N/D\\
\dot{\phi} &= \alpha,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
N =& 2J_{RW}M + 2bJ_{LT}m_h\sin\phi - \big(2I_hM + b^2m_h(M+m_t)\\
&+ bm_h(2aM\cos\phi + bm_h\cos(2\phi))\big)\alpha,\\
D =& m_h\left(2I + \left(2a^2+b^2\right)m_h\right) + 2m_t\left(I + \left(2a^2+b^2\right)m_h\right)\\
&+ 2a^2m_t^2 + bm_h\left(4aM\cos\phi + bm_h\cos(2\phi)\right),
\end{aligned}$$

and we regard the angular acceleration $\dot{\alpha}$ of the hinge angle to be a control input. It can easily be shown that $D > 0\ \forall\phi$.

4 HEADING STABILITY WITH LOCKED HEAD LINK

The authors of [2] consider the dynamics of a Chaplygin sleigh with a moving mass on top displaced from the sleigh's center of mass. If the mass were stationary, then such a system would be equivalent to the landfish for which the head link does not move, *i.e.*, $\dot{\phi} = 0$ and $\phi = \phi_0$. In [2], the system's locomotion was analyzed in terms of trajectory types, one of which is straight-line motion. Here we prove heading stability for a stationary link using Lyapunov theory.

For this case, the equations of motion (8) reduce to

$$\begin{aligned}
\dot{J}_{LT} &= c_1\dot{\theta}^2\\
\dot{J}_{RW} &= -\frac{c_1}{M}(J_{LT} + c_2\dot{\theta})\dot{\theta}\\
\dot{\theta} &= \frac{2}{D(\phi_0)}\left(MJ_{RW} + c_2J_{LT}\right),
\end{aligned} \tag{9}$$

where we define the constants

$$c_1 = aM + bm_h\cos\phi_0, \quad c_2 = bm_h\sin\phi_0.$$

Note that the evolution of J_{LT} depends on the sign of c_1. Regardless of the initial value of J_{LT}, its value over time is such that its sign eventually matches that of c_1, as long as $\dot{\theta}$ is not 0.

The system has an equilibrium where $\dot{\theta} = 0$, or

$$J_{RW} = -\frac{c_2}{M}J_{LT}. \tag{10}$$

Provided that the head link remains stationary with respect to the tail, this equilibrium is asymptotically stable, such that the landfish always approaches some constant heading for any set of initial conditions. Consider a Lyapunov function of the form

$$V = \left(J_{RW} + \frac{c_2}{M}J_{LT}\right)^2. \tag{11}$$

Clearly V is nonnegative everywhere and strictly equal to 0 at the equilibrium. Its time derivative is

$$\begin{aligned}
\dot{V} &= -2\frac{c_1}{M}J_{LT}\dot{\theta}\left(J_{RW} + \frac{c_2}{M}J_{LT}\right)\\
&= -\frac{4c_1}{MD}J_{LT}\left(\sqrt{M}J_{RW} + \frac{c_2}{\sqrt{M}}J_{LT}\right)^2.
\end{aligned} \tag{12}$$

The region around the equilibrium where $\dot{V} < 0$ is asymptotically stable. This holds as long as c_1J_{LT} is positive, which is ensured by the reduced equations (9). Thus, the landfish is guaranteed to converge to a constant orientation as $\dot{\theta}$ goes to 0.

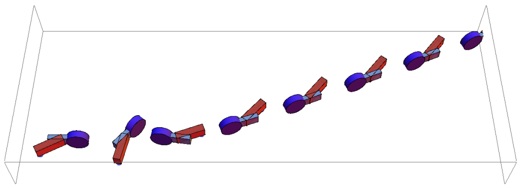

Figure 3. TRAJECTORY FOR LOCKED LINK CONFIGURATION.

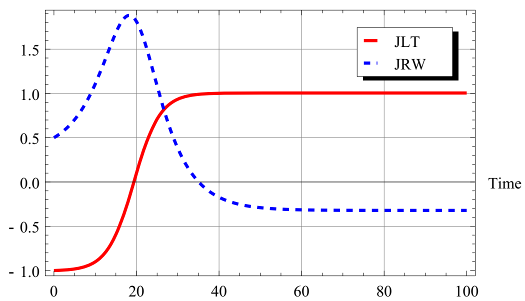

Figure 4. MOMENTA FOR LOCKED LINK CONFIGURATION.

Physically, c_1 is generally positive, unless $|\phi_0| > \frac{\pi}{2}$, meaning that the head link is more than a quarter turn away from the front orientation. Then c_1 can be negative if the head link's length b and mass m_h are sufficiently large. In this case, the system eventually ends up moving backward, as $J_{LT} < 0$.

5 OPEN-LOOP CONTROL SIMULATIONS

In the following simulations, we implement the above observations as well as a sinusoidal open-loop controller. We use the parameters $m_h = 2$, $m_t = 1$, $I_h = I_t = 1$, and $a = b = 1$.

For the first simulation, the head link remains stationary. The landfish starts at $(x_0, y_0, \theta_0) = (0, 0, \pi)$, and its initial conditions are $J_{RW0} = 0.5$, $J_{LT0} = -1$, and $\phi_0 = 0.5$. Figure 3 shows the system's trajectory at evenly timed snapshots as it locomotes. It starts out moving backward, but it eventually turns around and ends up moving forward instead. The instability of its initial motion pushes the landfish into its new pose, and it asymptotically approaches a positive heading angle. Figure 4 shows the changes in both J_{LT} and J_{RW} when this occurs.

Next we consider a sinusoidal input to the controller, such that $\dot{\alpha} = -\sin t$. Now that we are actuating the link, the system is able to locomote even starting from rest. If we use the initial conditions $J_{RW0} = J_{LT0} = 0$, $\phi_0 = 0$, and $\alpha_0 = 1$, then the resultant link orientation profile is $\phi(t) = \sin t$.

As shown in Fig. 5, the landfish's orientation oscillates opposite the motion of its head link. This particular gait thus replicates fishlike swimming, as the oscillation of the head and tail links with respect to each other generates forward propulsion.

Figure 5. TRAJECTORY FOR SINUSOIDAL CONTROL INPUT.

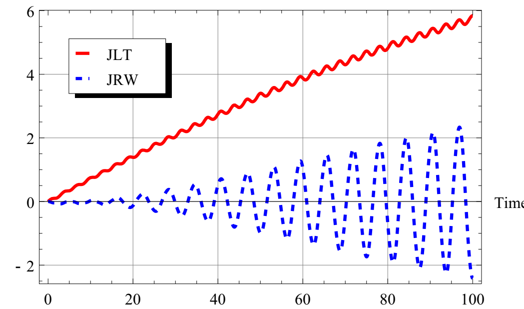

Figure 6. MOMENTA FOR SINUSOIDAL CONTROL INPUT.

The system also achieves an offset in the heading away from its initial orientation of $\theta_0 = 0$. Figure 6 shows its momenta, both of which are increasing in magnitude over time.

6 CLOSED-LOOP CONTROL

As with the systems of [3,4], we can achieve a desired heading using single-parameter proportional control applied to the input $\dot{\alpha}$. If we wish to converge to a heading of θ_d, we dictate

$$\dot{\alpha} = k(\theta - \theta_d), \tag{13}$$

where k is a tunable control gain parameter. Here, we choose k to be positive; by accelerating the head link in the direction of the heading offset, the system tends to rotate in the opposite direction back toward θ_d. In the simulation that follows, the system starts from rest and has an initial orientation $\theta_0 = 0.5$. We choose $\theta_d = 0$ as our desired steady-state heading.

Figure 7 shows the generated trajectory with $k = 5$. The heading quickly stabilizes to 0, while the head link continues to spin as the system moves forward. Fig. 8 shows the momenta; the linear component J_{LT} approaches a constant, while J_{RW} oscillates with a constant magnitude as the head's rotation stabilizes.

We use the same conditions to generate Fig. 9, except with $k = 10$. Not only does the system stabilize sooner with a higher gain parameter, but the magnitudes of both momenta components are higher. This is analogous to the Chaplygin beanie, in which the final value of J_{LT} is determined by k.

In the previous simulations, the landfish acquires a positive joint velocity while stabilizing its heading. It may be desirable to control this parameter as well; for example, a physical system

Figure 7. HEADING CONTROL DEMONSTRATION WITH $k = 5$.

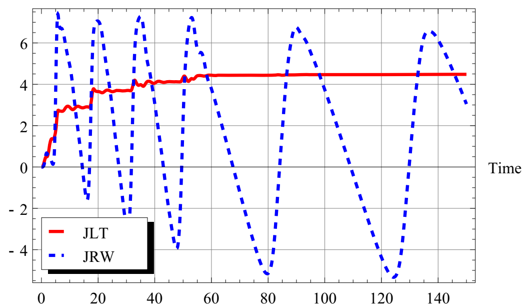

Figure 8. HEADING CONTROL MOMENTA EVOLUTION FOR $k = 5$.

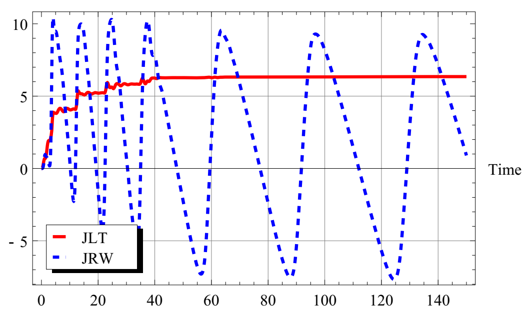

Figure 9. HEADING CONTROL MOMENTA EVOLUTION FOR $k = 10$.

would benefit without having to constantly actuate its motors. We thus propose a novel two-step control approach. In the first step, the system overshoots the desired heading, approaching a nonzero joint velocity as before. In the second step, the system corrects back to the desired velocity with a different controller gain, at the same time reverting its joint velocity to zero.

While we do not yet have analytical results allowing us to easily determine the optimal overshoot angle and gain parameters, we implement this technique experimentally in Fig. 10. With the same initial conditions as before, we use a gain parameter of $k_1 = 10$ and aim for $\theta_d = -0.3$. When the landfish has stabilized after sufficient time, we switch the gain parameter to $k_2 = 1$ and aim for $\theta_d = 0$. As shown, the landfish successfully achieves both the desired heading and joint velocity of zero.

7 CONCLUSIONS AND FUTURE WORK

In this paper, we have presented the landfish, a novel and nonlinear system extending the Chaplygin sleigh. By controlling its joint rotation in different ways, the landfish can achieve different trajectories, including those of fishlike locomotion. Like the Chaplygin beanie, the landfish's heading can also be stabilized using proportional control, and we have applied a controller to account for joint velocity as well.

The fact that the landfish can locomote like a fish from periodic joint inputs suggests that one can potentially apply gait analysis to achieve motion planning for various desired trajectories. In terms of closed-loop control, the next logical step would be to analytically prove heading stability and momenta convergence, perhaps for a set of parameters satisfying certain conditions. Finally, the two-step controller approach can be formally presented with analytical guarantees, and perhaps generalized to previous systems such as the Chaplygin beanie as well.

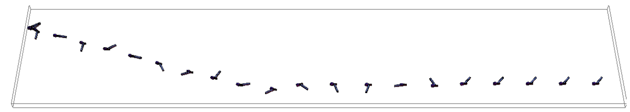

Figure 10. TRAJECTORY OF TWO-STEP CONTROLLER.

ACKNOWLEDGMENT

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