Path-Connectivity of the Free Space

Alberto Rodriguez and Matthew T. Mason

***Abstract*—This paper revisits the notion of free configuration space and reviews some of its path-connectivity related properties. Literature on motion planning reveals at least three different definitions for the free configuration space of a robot in the presence of obstacles. This paper shows that, assuming regularity of both object and obstacles, those three definitions are equivalent. We show the three definitions regularize the free space and therefore prevent the existence of "thin bits", or low dimensional strata. The paper concludes discussing a series of properties regarding the existence and smoothability of contact-free paths between pairs of configurations.**

***Index Terms*—Configuration space, Free Space, Connectivity, Path planning.**

I. Introduction

The configuration space of a system, a concept borrowed from classical mechanics, refers to the space of its possible states. Lozano-Perez [1] was the first to explicitly use the concept in the context of collision-free motion planning. Over the years it has proven to be a useful abstraction to unify diametrically different problems, such as the *piano mover's problem*, where the objective is to find a collision-free path between two configurations, or the *caging problem*, where the objective is to void the existence of such path.

From the point of view of the configuration space, most variations of the problem of planning under constraints can be abstracted to finding either the existence or the absence of a path between pairs of configurations. Hence, the topology of the free space is key to the configuration space approach. In this work, we gather and contribute to a series of results in the literature relating to the path-connectivity of the free space.

Motion planning research was considerably unified following Latombe's book [2]. Still, a brief overview of the literature shows different lines of work based upon different definitions of free space. We begin this communication reviewing three definitions and putting them in context (Sect. III). We then discuss the conditions under which they are equivalent (Sect. IV), and how they all effectively remove "thin bits" from the free space (Sect. V). We finish reviewing properties regarding the existence and smoothability of paths between pairs of configurations in the interior of the free space (Sects. VI and VII).

II. Basic Topological Concepts

In this paper we make use of the basic topological concepts of open set, closed set, complement ($\setminus$), interior (int), closure (cl), exterior (ext), boundary (∂), regularity, and path-connectedness. Here we give a brief description. For a more complete introduction we refer to chapters two and four of Lee's book on topological manifolds [3].

Let $(\mathcal{M}, d)$ be a metric space. The *complement* of a subset U is the set $\mathcal{M} \setminus U = \{x \in \mathcal{M} : x \notin U\}$. We say a subset U is *open* if for every point $x \in U$, there is a ball $B_\epsilon(x) = \{y \in \mathcal{M} : d(x, y) < \epsilon\}$ such that $B_\epsilon(x) \subset U$. Conversely, a subset U is *closed* if and only if its complement $\mathcal{M} \setminus U$ is open.

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(a) (b) (c)

Fig. 1. (a) Workspace of a circular robot with two rectangular obstacles in the workspace. (b) $\mathcal{C}_{obs}$ voids contact between robot and obstacles. (c) $\mathcal{C}^\circ_{obs}$ allows contact and does not prevent the robot from crossing the gap in between the obstacles.

The *interor* of $U \subset \mathcal{M}$ is the largest open set contained in U, and its *closure* the smallest closed set containing U:

$$\begin{aligned} \text{int}[U] &= \bigcup \{A \subset \mathcal{M} : A \subset U \text{ and } A \text{ open}\} \\ \text{cl}[U] &= \bigcap \{A \subset \mathcal{M} : U \subset A \text{ and } A \text{ closed}\} \end{aligned}$$

The *exterior* of a set U is $\text{ext}[U] = \mathcal{M} \setminus \text{int}[U]$, and its *boundary* $\partial U = \mathcal{M} \setminus (\text{int}[U] \cup \text{ext}[U])$. So that, for any set U, the whole space $\mathcal{M}$ is can be decomposed into the disjoint union $\text{int}[U] \cup \partial U \cup \text{ext}[U]$.

We say a set U is *regular* if $\text{cl}[\text{int[U]}] = U$. In this paper we informally call "thin bits" to the difference between U and its regularized version $\text{reg}\,[U] = \text{cl}[\text{int[U]}]$. Finally, a set U is *path-connected* if for every pair of points $x \in U$, $y \in U$, there is a continuous map $\alpha : [0, 1] \mapsto U$ with $\alpha(0) = x$ and $\alpha(1) = y$.

III. Free Congiguration Space

Let $\mathcal{A}$ be a robot and $\mathcal{O}$ an obstacle, both closed subsets of the workspace $\mathcal{W} \simeq \mathbb{R}^n$. We note by $\mathcal{C}$ the *configuration space* of the robot. For each $q \in \mathcal{C}$, $\mathcal{A}_q$ is a closed subset of $\mathcal{W}$. The *configuration space obstacle* $\mathcal{C}_{obs}$ is the set of configurations q where the closed sets $\mathcal{A}_q$ and $\mathcal{O}$ collide. The set of *admissible configurations* is the complement of $\mathcal{C}_{obs}$.

The discrepancies between definitions of free space originate with the different notions of collision. Depending on whether contact between object and obstacle is allowed or not, the configuration space obstacle can be defined as:

$$\mathcal{C}_{obs} = \{q \in \mathcal{C} \mid \mathcal{A}_q \cap \mathcal{O} \neq \varnothing\} \quad (1)$$

$$\mathcal{C}^\circ_{obs} = \{q \in \mathcal{C} \mid \text{int}\,[\mathcal{A}_q] \cap \text{int}\,[\mathcal{O}] \neq \varnothing\} \quad (2)$$

$\mathcal{C}_{obs}$ is closed and defines contact configurations as colliding. $\mathcal{C}^\circ_{obs}$ is open and does allow contact between robot and object. The *free configuration space* is in general conceived as the complement of the configuration space obstacle. Unfortunately, neither of both induced free spaces are ideal. On one hand $\mathcal{C} \setminus \mathcal{C}_{obs}$ is open and consequently not compact, which is not suitable to formulate optimization problems. On the other hand, $\mathcal{C} \setminus \mathcal{C}^\circ_{obs}$ is closed and allows contact, but, as illustrated in Fig. 1, it is not restrictive enough to void the existence of "thin bits".

Both issues can be simultaneously addressed with small modifications to the "nominal" definition of free space. We find in the literature at least three different definitions that attempt to do it:

$$\mathcal{C}_{free_1} = \text{cl}\,[\mathcal{C} \setminus \mathcal{C}_{obs}] \quad (3)$$

$$\mathcal{C}_{free_2} = \mathcal{C} \setminus \text{int}\,[\mathcal{C}_{obs}] \quad (4)$$

$$\mathcal{C}_{free_3} = \text{cl}\,[\text{int}\,[\mathcal{C} \setminus \mathcal{C}^\circ_{obs}]] \quad (5)$$

They all allow contact between robot and object and, as we shall see in Section V, they also correctly regularize the free space by eliminating "thin bits".

$\mathcal{C}_{free_1}$ is the most extended definition of the three. It is based on the original definition by Lozano-Perez [1] and has been adopted by most of the literature on motion planning [2], [4]–[6]. It is sometimes referred to as semi-free space.

$\mathcal{C}_{free_2}$ is the conventional definition used in configuration space approaches to caging [7]–[10]. The same way that the caging problem can be seen as dual to the piano mover's problem, we shall see in Proposition 1 that $\mathcal{C}_{free_2}$ can also be seen as dual to $\mathcal{C}_{free_1}$.

$\mathcal{C}_{free_3}$ allows contact between object and obstacles by only considering their interiors. We can trace back the use of $\mathcal{C}_{free_3}$ to early studies of the motion of objects in contact [11], and early work on trajectory planning for mobile robots [12]–[14]. The definition was later adopted by part of the field of discrete and computational geometry [15], [16].

Laumond proposed [13] a limit process to restrict the permissive definition of the free space in $\mathcal{C} \setminus \mathcal{C}^{\circ}_{obs}$ and proved the output equivalent to $\mathcal{C}_{free_3}$. In the same work, Laumond studies the existence of contact-free paths in that free space. Inspired by that analysis we extend the study of the existence of contact-free paths to all three definitions in Sect. VI.

IV. Equivalence Between Definitions

In the following proposition we state sufficient conditions for the three definitions of free space to be equivalent.

Proposition 1 (Equivalences between definitions of free space): The next equivalences hold:

i. $\mathcal{C}_{free_1}$ and $\mathcal{C}_{free_2}$ are always equivalent.
ii. $\mathcal{C}_{free_3}$ is equal to $\mathcal{C}_{free_1}$ and $\mathcal{C}_{free_2}$ if both the robot $\mathcal{A}$ and the obstacle $\mathcal{O}$ are regular sets.

Proof: The first equivalence is consequence of the duality between the closure and interior operators (Lemma 2 in the Appendix).

To prove the second equivalence, we use again the dual relation between closure and interior on the definition of $\mathcal{C}_{free_3}$: $\text{cl}\,[\text{int}\,[\mathcal{C} \setminus \mathcal{C}^{\circ}_{obs}]] = \text{cl}\,[\mathcal{C} \setminus \text{cl}\,[\mathcal{C}^{\circ}_{obs}]] = \mathcal{C} \setminus \text{int}\,[\text{cl}\,[\mathcal{C}^{\circ}_{obs}]]$. This is equal to $\mathcal{C}_{free_2}$ if and only if $\text{int}\,[\text{cl}\,[\mathcal{C}^{\circ}_{obs}]] = \text{int}\,[\mathcal{C}_{obs}]$.

We find in Latombe [2] that if both $\mathcal{A}$ and $\mathcal{O}$ are regular, then $\mathcal{C}_{obs} = \text{cl}\,[\mathcal{C}^{\circ}_{obs}]$, which concludes the proof. ■

Thanks to the equivalences in Proposition 1, we will name $\mathcal{C}_{free}$ indistinctively to either $\mathcal{C}_{free_1}$ or $\mathcal{C}_{free_2}$, and $\mathcal{C}^{\circ}_{free}$ to $\mathcal{C}_{free_3}$.

If either the robot $\mathcal{A}$ or the obstacle $\mathcal{O}$ are not regular, the equivalence in the second part of Proposition 1 does not necessarily hold, as illustrated in Fig. 2. Although both definitions allow contact, $\mathcal{C}_{free}$ is in general more restrictive than $\mathcal{C}^{\circ}_{free}$.

(a) (b) (c)

Fig. 2. (a) Workspace of a circular robot and a non-regular obstacle. (b) $\mathcal{C}_{free}$. (c) $\mathcal{C}^{\circ}_{free}$. These definitions of the free space are not equivalent.

V. Regularity of the Free Space

Proposition 2 (Regularity of the free space): Both $\mathcal{C}_{free}$ and $\mathcal{C}^{\circ}_{free}$ are regular subsets of $\mathcal{C}$, and consequently do not have "thin bits".

Proof: By definition, a set $\mathcal{S}$ is regular iff $\text{cl}\,[\text{int}\,[\mathcal{S}]] = \mathcal{S}$. Using Lemma 3 from the Appendix and the fact that $\mathcal{C}_{obs}$ is closed, we have:

$$\begin{aligned}\text{cl}\,[\text{int}\,[\mathcal{C}_{free}]] &= \text{cl}\,[\text{int}\,[\text{cl}\,[\mathcal{C} \setminus \mathcal{C}_{obs}]]] \\ &= \text{cl}\,[\text{int}\,[\text{cl}\,[\text{int}\,[\mathcal{C} \setminus \mathcal{C}_{obs}]]]] \\ &= \text{cl}\,[\text{int}\,[\mathcal{C} \setminus \mathcal{C}_{obs}]] \\ &= \text{cl}\,[\mathcal{C} \setminus \mathcal{C}_{obs}] = \mathcal{C}_{free}\end{aligned}$$

Consequently, $\mathcal{C}_{free}$ is regular. $\mathcal{C}^{\circ}_{free}$ is also regular as direct consequence of Lemma 3. ■

By virtue of Propositions 1 and 2, all three original definitions of free space avoid the existence of "thin bits".

VI. Path-connectivity of the Free Space

The following proposition shows that between two configurations in the same path-connected component of the free space, there is always a path that avoid contact with obstacles except at isolated configurations. The result was first proven by Laumond [13] for the specific case of $\mathcal{C}^{\circ}_{free}$.

Proposition 3 (Path-connectivity of the free space): If the free space is regular, any two configurations c_1 and c_2 in the same path-connected component can be connected by a path $\gamma(t)$ in the interior of the free space except at isolated points.

Proof: Let $\mathcal{F}$ denote the path-connected component of the free space to which both configurations c_1 and c_2 belong. If $\text{int}\,[\mathcal{F}]$ is path-connected, there is a trajectory $\gamma(t)$ in $\text{int}\,[\mathcal{F}]$ that connects them.

Let's suppose then that $\text{int}\,[\mathcal{F}]$ decomposes in two path-connected components $\text{int}\,[\mathcal{F}] = \mathcal{F}_1 \cup \mathcal{F}_2$ with $c_1 \in \mathcal{F}_1$ and $c_2 \in \mathcal{F}_2$ (note that both $\mathcal{F}_1$ and $\mathcal{F}_2$ must be open). By hypothesis, $\mathcal{F}$ is regular and hence:

$$\mathcal{F} = \text{cl}\,[\text{int}\,[\mathcal{F}]] = \text{cl}\,[\mathcal{F}_1 \cup \mathcal{F}_2] = \text{cl}\,[\mathcal{F}_1] \cup \text{cl}\,[\mathcal{F}_2]$$

which implies that $\text{cl}\,[\mathcal{F}_1] \cap \text{cl}\,[\mathcal{F}_2] \neq \varnothing$ (otherwise $\mathcal{F}$ could not be path-connected). Let $y \in \partial\mathcal{F}_1 \cap \partial\mathcal{F}_2$ and V an open neighborhood of y. The set $V \cap \mathcal{F}_1$ contains a path γ_{y_1} from y to t_1, a configuration inside $\mathcal{F}_1$. Because $\mathcal{F}_1$ is path-connected, there is a path γ_1 from t_1 to c_1 inside $\mathcal{F}_1$. Call $\gamma_{\mathcal{F}_1}$ to the concatenation of γ_1 with γ_{y_1}, a path from y to c_1 in the interior of the free space, except at y and possibly at c_1. Equally, we can construct a path $\gamma_{\mathcal{F}_2}$ from y to c_2 also in the interior of the free space. The concatenation of $\gamma_{\mathcal{F}_1}$ with $\gamma_{\mathcal{F}_2}$ gives us the desired path.

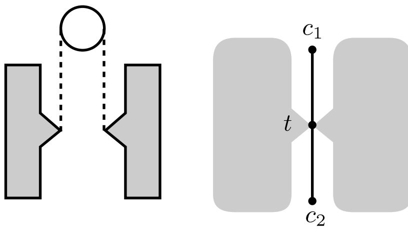
If $\text{int}\,[\mathcal{F}]$ happens to have more than two path connected components, we create paths from component to component, the same way as above for connecting c_1 to c_2 through the puncture points y in the boundary between path-connected components of $\text{int}\,[\mathcal{F}]$. ■

Figure 3 shows an example of a free space where path connectivity requires contact with the boundary. The following proposition shows that the degeneracy in Fig. 3b never happens when the free space has the structure of a topological manifold.

Proposition 4 (Path-connectivity of the free space for manifolds): If the free space is a manifold with boundary, any two configurations c_1 and c_2 in the same path-connected component can be connected by a path $\gamma(t)$ in the interior of the free space.

Proof: We recreate the exact same construction as in the proof of Proposition 3. Suppose now that $\text{int}\,[\mathcal{F}]$ has two path-connected components. Let t be the puncture point in $\partial\mathcal{F}_1 \cap \partial\mathcal{F}_2$ and V an open neighborhood around it.

By hypothesis, the free space is a manifold with boundary and, as such, every point must be locally Euclidean (homeomorphic to an m-dimensional open ball if the point is in the interior or to an m-dimensional closed half space if the point is in the boundary). In particular, the interior of the neighborhood $V \cap \mathcal{F}$ becomes path-disconnected, which contradicts the fact that $\mathcal{F}$ must be locally Euclidean at t. Consequently, $\text{int}\,[\mathcal{F}]$ must be path-connected. ■



(a) (b)

Fig. 3. Example of a path that "needs" to touch the obstacle at t in order to go from c_1 to c_2 (suppose the outside is blocked by some other obstacle). Notice that the free space is not a manifold precisely because of t.

VII. Smooth Path-Connectivity of the Free Space

When searching for paths, it is sometimes desirable to constrain the search to smooth instead of just continuous paths. Lemma 1, proven by Latombe [2], allows us to guarantee the existence of smooth ϵ-approximations of continuous paths between pairs of configurations.

Lemma 1: Any path $\gamma(t)$ in the interior of the free space is homotopic to a piecewise smooth path $\tilde{\gamma}(t)$ also in the interior of the free space that is arbitrarily close to it:

$$|\gamma(t) - \tilde{\gamma}(t)| \leq \epsilon \quad \forall\ t \in [0,1]$$

The construction of the proof uses a finite covering of $\gamma(t)$ with open sets of a chart. A smooth approximation of $\gamma(t)$ on each set of the covering completes the proof. Note that we can chose to use the Weierstrass approximation theorem [17] to approximate $\gamma(t)$ by a polynomial on each set of the covering. The piecewise smooth approximation becomes then a semi-algebraic approximation.

The following is a corollary of Propositions 3 and 4 for smooth paths:

Corollary 1 (Smooth path-connectivity of the free space): If the free space is regular, any two configurations c_1 and c_2 in the same path-connected component can be connected by a piecewise smooth path $\gamma(t)$ in the interior of the free space, except at isolated points. If the free space is a manifold with boundary, the smooth path lies entirely in the interior of the free space.

Proof: Proposition 3 gives us a path that lives in the free space except at isolated configurations. We can then use Lemma 1 to piecewise approximate each one of the subpaths that connect those isolated configurations.

If the free space is a manifold, we use the same argument, but now with Proposition 4 and Lemma 1 to construct the piecewise smooth path, now entirely in the interior of the free space. ■

VIII. Conclusions

A review of the literature on motion planning reveals at least three different definitions for the free space of a robot moving among obstacles. We show in this paper that when both robot and obstacle are regular subsets of the workspace, all three definitions are equivalent. Combining topological operators interior and closure, all three definitions allow contact and regularize the free space, eliminating "thin bits".

We have further shown that regularity of the free space is enough to guarantee that any two configurations of the robot in the same path-connected component are connected by a piecewise smooth path that lives in the interior of the free space except at isolated configurations. If the free space is also a manifold, there is a smooth path connecting them that lies completely in the interior of the free space.

Correctly formulating a problem in the context of motion planning usually requires the free space to be compact, i.e., requires to allow contact with obstacles, since any optimization-driven approach might not have a solution otherwise. On the other hand, depending on the application, contact with obstacles might not be desirable. If it is the case that the free space is regular, one way to approach it is to solve the motion planning problem allowing contact, and if the solution does involve contact, we can always find an equivalent contact-free path.

Being able to restrict the analysis to contact-free paths is also useful in the context of caging [10]. The analysis of existence of path-connectivity in caging problems becomes easier if we only have to consider contact-free paths, since they allow some play before stepping outside the free space.

Appendix
Topology Review

In this appendix we review two key properties of the topological operators *closure* and *interior* used in this work.

Lemma 2 (Duality of closure and interior): The operators $\mathrm{cl}[\cdot]$ and $\mathrm{int}[\cdot]$ are dual in the sense that, for any given subset A of a topological space X:

$$\text{i. } \mathrm{cl}[A] = X \setminus \mathrm{int}[X \setminus A]$$
$$\text{ii. } \mathrm{int}[A] = X \setminus \mathrm{cl}[X \setminus A]$$

Proof: We prove the first identity. The second one is obtained by replacing A by $X \setminus A$.

By definition, $\mathrm{cl}[A] = X \setminus \mathrm{int}[X \setminus A]$ if and only if $X \setminus \mathrm{int}[X \setminus A]$ is the smallest closed set that contains A, i.e., if and only if for all closed sets B inside A, it is true that $X \setminus \mathrm{int}[X \setminus A] \subseteq B$.

Let's suppose that, contrary, there is a closed set B such that:

$$A \subset B$$
$$B \subset X \setminus \mathrm{int}[X \setminus A]$$

From the first condition we derive that $X \setminus B \subset X \setminus A$. The second condition tells us that $X \setminus B \supset \mathrm{int}[X \setminus A]$. By definition $\mathrm{int}[X \setminus A]$ is the largest open set contained in $X \setminus A$. In particular, $X \setminus B$ is open and contained in $X \setminus A$, therefore by combining both conditions we get:

$$\setminus B \subset \mathrm{int}[X \setminus A] \subset X \setminus B$$

which is self contradictory and concludes the proof. ■

Lemma 3: The regularizer operator $\mathrm{reg}[\cdot] = \mathrm{cl}[\mathrm{int}[\cdot]]$ is idempotent.

Proof: We want to show that $\mathrm{reg}[\mathrm{reg}[A]] = \mathrm{reg}[A]$ for any given subset A. For that we make use of Theorem 6 in Kuratowski's analysis of the topological closure operator [18]:

$$\mathrm{cl}[X \setminus \mathrm{cl}[X \setminus \mathrm{cl}[X \setminus \mathrm{cl}[B]]]] = \mathrm{cl}[X \setminus \mathrm{cl}[B]] \qquad (6)$$

valid for any subset B of a topological space X. If we particularize (6) for $B = X \setminus A$ we get:

$$\mathrm{cl}[X \setminus \mathrm{cl}[X \setminus \mathrm{cl}[X \setminus \mathrm{cl}[X \setminus A]]]] = \mathrm{cl}[X \setminus \mathrm{cl}[X \setminus A]] \qquad (7)$$

Using the duality Lemma 2 to the left side of (7) we get:

$$\begin{aligned}\mathrm{cl}[X \setminus \mathrm{cl}[X \setminus \mathrm{cl}[X \setminus \mathrm{cl}[X \setminus A]]]] &= \mathrm{cl}[\mathrm{int}[\mathrm{cl}[X \setminus \mathrm{cl}[X \setminus A]]]] \\ &= \mathrm{cl}[\mathrm{int}[\mathrm{cl}[\mathrm{int}[A]]]] \\ &= \mathrm{reg}[\mathrm{reg}[A]]\end{aligned}$$

If now we use the duality Lemma 2 to the right side of (7) we get:

$$\mathrm{cl}[X \setminus \mathrm{cl}[X \setminus A]] = \mathrm{cl}[\mathrm{int}[A]] = \mathrm{reg}[A]$$

which concludes the proof. ■

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