Two Finger Caging: Squeezing and Stretching

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Abstract: This paper studies the problem of placing two point fingers to *cage* a mobile rigid body in a Euclidean space of arbitrary dimension. (To *cage* an object is to arrange obstacles so that all motions of the mobile body are bounded.) This paper shows that if a compact connected contractible object is caged by two points, then it is either *stretching caged* or *squeezing caged* or both, where *stretching caged* means the body is trapped even if the point fingers are given the freedom of moving apart, and *squeezing caged* means the the body is trapped even if the fingers are given the freedom of moving closer. This result generalizes a previous result by Vahedi and van der Stappen [18] which applied to two points trapping a polygon in the plane. Our use of a topological approach led to the generalization, and may lead to further generalizations and insights.

1 Introduction

A cage is an arrangement of obstacles that bounds the collision-free paths of some object. Caging is interesting for two reasons. First, caging an object is a way to manipulate it without immobilizing it. Second, even if an immobilizing grasp is needed/preferred over a cage, the cage may provide a useful waypoint to the immobilizing grasp. From some cages a *blind policy* exists that achieves an immobilization while preserving the cage.

Caging is weaker than immobilizing. While the objective of an immobilizing grasp might be to precisely locate an object relative to the hand, the objective of a cage is just to guarantee that the object is within the reach of the manipulator and cannot escape. By weakening the objective, the ultimate task may be easier both in theory and in practice.

Caging an object with point fingers assumes that the fingers are rigidly fixed relative to the hand and to each other. Vahedi and van der Stappen[18]

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recently introduced a variation where the fingers are allowed some relative motion while the object remains caged. An object is *squeezing caged* if all its motions are bounded, even if the fingers are allowed to move while not increasing the initial separation between them. Similarly, the object is *stretching caged* if it cannot escape, even if the fingers are allowed to move while not decreasing the initial separation between them. Vahedi and van der Stappen showed that any cage of a planar polygon by two disk fingers is either a squeezing cage, a stretching cage, or both.

This paper extends Vahedi and van der Stappen's result to include arbitrary compact connected contractible objects in Euclidean spaces of arbitrary dimension. Thus the squeezing and stretching caging characterization becomes a fundamental property of the configuration space of a two fingered manipulator.

2 Related Work

The earliest mathematical work on trapping objects was by Besicovitch in 1957 [1] and Shephard in 1965 [13]. Both worked on the problem first posed by Besicovitch as a *contest problem* to undergraduates, of trapping a sphere with a net. However, it was not until 1990 that Kuperberg [3] posed a formal definition of the 2D caging problem:

> "Let P be a polygon in the plane, and let C be a set of k points which lies in the complement of the interior of P. The points capture P if P cannot be moved arbitrarily far from its original position without at least one point of C penetrating the interior of P. Design an algorithm for finding a set of capturing points for P."

Since then, there have been several approaches to the problem, from different perspectives and with different goals. Rimon and Blake [10, 11] introduced the notion of a *caging set*: the maximal connected set of caging configurations that contains a given grasping configuration. They applied Morse Theory to the case of 1-parameter two-fingered grippers to show that the limit configurations where the cage is broken correspond to equilibrium grasps of the object. Later, Davidson and Blake [2] extended the result to 1-parameter 3-fingered planar grippers.

Sudsang and Ponce [15, 16] proposed and studied the application of caging to motion planning for three disc-shaped planar robots manipulating an object. They provided a geometrical method to compute conservative approximations of the so called *Inescapable Configuration Space regions*. They also analyzed in-hand manipulation using caging [17, 14].

Pereira, Campos and Kumar [9] applied caging to decentralized multirobot manipulation. They used the geometrical description of the robots to develop a conservative, on-line and decentralized test to maintain cageness.

Vahedi and van der Stappen [18] formalized squeezing and stretching caging to cage polygonal objects with two disc-shaped fingers, and used that idea to develop the first complete algorithm to compute the entire caging set of two fingers. Their algorithm generates a graph structure in the configuration space of the fingers that finds all caging grasps in $O\left(n^2 \log n\right)$ and handles cageness queries in $O(\log n)$.

3 Preliminary Concepts

3.1 Configuration Space

Assume the *workspace* to be $\mathbb{R}^d$, and let the *manipulator* be a set of position-controlled point *fingers* $p_1 \dots p_n$ in $\mathbb{R}^d$. Let P_i be the configuration space of finger p_i, and let $M = P_1 \times \dots \times P_n$ be the configuration space of the manipulator ($\dim M = n \cdot d$).

We assume the object O is a compact region of the workspace. It induces an obstacle for each of the fingers, and therefore for the manipulator. Let O_i be the obstacle induced for finger i in P_i, and let O^M be the obstacle induced for the manipulator in M. We can decompose the manipulator obstacle:

$$O^M = \{ (p_1 \dots p_n) \in M \mid \exists p_i \in O \} = \bigcup_{i=1}^{n} \{ (p_1 \dots p_n) \mid p_i \in O \} = \bigcup_{i=1}^{n} O_i^M \tag{1}$$

where O_i^M is the obstacle induced in M by the interaction of object O with finger i.

Following the convention of [12, 18] we define the *free space of the manipulator* to be $M^{\text{free}} = M \setminus \text{Int}\left(O^M\right)$, where $\text{Int}\left(O^M\right)$ is the interior of O^M as a subset of $\mathbb{R}^{n \cdot d}$. We define the configurations in M^{free} to be the *admissible* configurations of the manipulator. This definition induces a free space in the configuration space of finger i, $P_i^{\text{free}} = P_i \setminus \text{Int}(O)$.

Note that there are alternative, more liberal, definitions of admissible configurations [5, 4]. The advantage of the stricter convention is that the free space is *regular*—it is the closure of its interior. Consequently it has no "thin bits" (Fig. 1). We will skip the proof to conserve space. By [5], regularity of the free space enables the following proposition:

Proposition 1 (Connectivity of the Free Space). *For any two configurations in the same connected component of the free space, an admissible connecting path exists, and lies in the interior of the free space except possibly at some isolated points.*

Proposition 1 implies that if two configurations of the manipulator are in the same connected component of the free space, they can be joined by a path that avoids contact with the object O except at isolated configurations. This property is essential to prove the main result in Section 5.3.

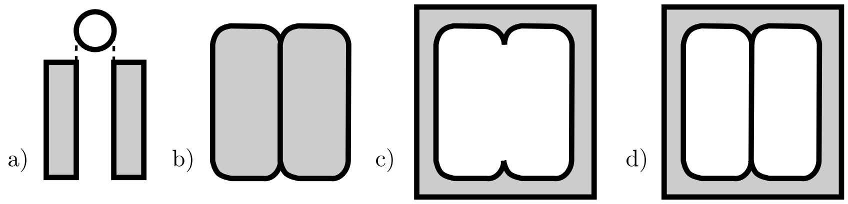


Fig. 1. a) Workspace: two obstacles and a moving object. b) Cspace obstacles. c) Free space. d) A more liberal definition of free space with unwanted thin bits.

3.2 Paths in the Configuration Space

Given a configuration c of the manipulator in M^{free}, a *closed path based at c* refers to a parameterized curve $\alpha : [0,1] \longrightarrow M^{\text{free}}$ with $\alpha(0) = \alpha(1) = c$. A *contractible path* is a closed path that is path homotopic to a point in M^{free}.

Contractibility of a path, then, implies the existence of a continuous map H, called a *homotopy of paths*:

$$H : [0,1] \times [0,1] \longrightarrow M^{\text{free}} \tag{2}$$

such that $H(t,0) = \alpha(t)$ and $H(t,1) = H(0,s) = H(1,s) = c$, for all $t, s \in [0,1]$.

The next proposition, with proof in appendix A, relates the contractibility of paths in M^{free} to the contractibility of the individual fingers' paths. First we define Π_i, the *natural projection* from the configuration space of the manipulator to the configuration space of finger i:

$$\begin{array}{rcl} \Pi_i : \quad M & \longrightarrow & P_i \\ (p_1 \ldots p_n) & \longrightarrow & p_i \end{array} \tag{3}$$

Proposition 2 (Characterization of contractible paths). *A closed path α at c is contractible in M^{free} if and only if for each finger i, $\Pi_i(\alpha)$ describes a contractible path in P_i^{free}.*

4 Caging

4.1 Introduction to Caging

To formalize the definition of caging proposed by Kuperberg [3] we could consider an object to be caged if and only if the object configuration lies in a compact connected component of its free space.

However, it is simpler and equivalent to consider the object to be fixed, and instead study the rigid motions of the manipulator, yielding the definition:

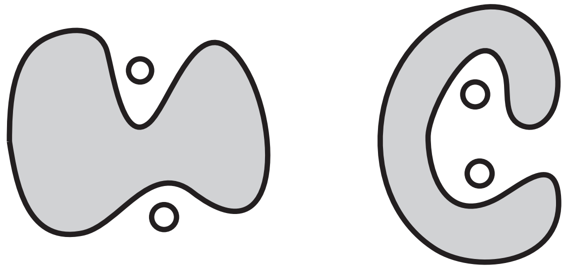


Fig. 2. Examples of squeezing (left) and stretching (right) caging configurations.

Definition 1 (Caging Configuration). *Let M_c be the set of all configurations with the same pairwise finger distances as c. A caging configuration is a configuration c of the manipulator that lies in a compact connected component of $M^{free} \cap M_c$.*

Hence, an object is caged if and only if the manipulator is unable to escape from the object while preserving its shape.

In the case of a two fingered manipulator, the Euclidean distance between the fingers is the only constraint that defines M_c. Let the map $r : M \longrightarrow \mathbb{R}$ be that distance. The set M_c is then defined as:

$$M_c = \{ q \in M \mid r(q) = r(c) \} \tag{4}$$

4.2 Squeezing and Stretching Caging

Intuitively, the manipulator is in a *squeezing* (*stretching*) caging configuration if the object cannot escape, even by allowing the fingers to move closer (separate), Fig. 2. The definition can be formalized in a similar way as in the case of caging.

Let $\underline{M}_c$ and $\overline{M}_c$ be the sets:

$$\underline{M}_c = \{ q \in M \mid r(q) \leq r(c) \} \tag{5}$$
$$\overline{M}_c = \{ q \in M \mid r(q) \geq r(c) \} \tag{6}$$

Then we define:

Definition 2 (Squeezing Caging configuration). *Configuration c of the manipulator that lies in a compact connected component of $M^{free} \cap \underline{M}_c$.*

Definition 3 (Stretching Caging configuration). *Configuration c of the manipulator that lies in a compact connected component of $M^{free} \cap \overline{M}_c$.*

The main objective of this work is to show that all caging configurations are either squeezing caging, stretching caging or both.

5 The Squeezing and Stretching Caging Theorem

5.1 The Result

Theorem 1 (Squeezing-Stretching Caging). *Given a two finger caging configuration of a compact connected contractible object in* $\mathbb{R}^d$*, it is squeezing caging, stretching caging or both.*

We will prove the contrapositive. Suppose that a certain two finger configuration c is neither squeezing nor stretching caging. That means there is an escape path $\underline{\alpha}$ if squeezing is permitted, and there is also an escape path $\overline{\alpha}$ if stretching is permitted. We will use the existence of these two escape paths to construct a third escape path in M_c, establishing the noncageness of c. From a topological perspective, we can understand the constructed escape path as an *average* of the squeezing and stretching escaping paths.

The proof consists of two steps (Fig. 3):

1. Using the two escape paths we build a closed contractible curve in M^{free}, with the property that every crossing through M_c, except for c, is known to be noncaging.
2. When the contractible curve is actually contracted, the intersection with M_c will give a rigid body path from c to one of the known noncaging configurations.

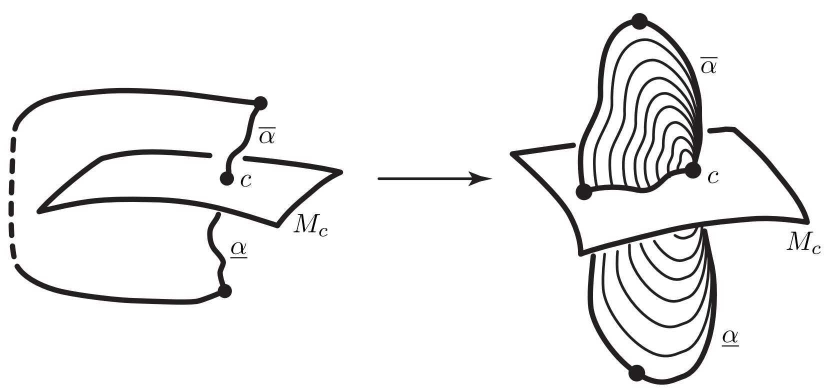


Fig. 3. The proof of theorem 1. (left) The contractible curve crossing M_c "at infinity". (right) Simplest case intersection between the contraction and M_c.

5.2 Building the Contractible Curve

The contractible curve must satisfy two requirements:

- All crossings of the curve through M_c, except for c, must be known to be noncaging.
- Both the curve and the contraction must live in M^{free}.

Let $B \subset \mathbb{R}^d$ be a ball containing O such that any placement of the fingers outside B is a noncaging configuration of the manipulator. We will say that any configuration with the fingers outside B is *at infinity*. Note that if O is compact, B always exists. For the case of two fingers we can choose B to be any circumscribing ball.

We can assume the escape paths $\underline{\alpha}$ and $\overline{\alpha}$ to terminate respectively at configurations $\underline{c}$ and $\overline{c}$ *at infinity*. Let $\underline{\alpha} \oplus \overline{\alpha}$ be the curve defined by the concatenation of both escaping paths. Closing $\underline{\alpha} \oplus \overline{\alpha}$ with an additional curve lying entirely outside B guarantees that all crossings of the complete curve through M_c will be noncaging. Hence we just need to see that always exists a curve β that closes $\underline{\alpha} \oplus \overline{\alpha}$ outside B in such a way that the complete closed curve is contractible.

By proposition 2, we can construct the completion of the curve independently for each finger. As long as each projected path $\Pi_i(\underline{\alpha} \oplus \overline{\alpha})$ is closed in a contractible way in P_i^{free}, the curve in M and its contraction will be in M^{free}, satisfying the second requirement. Let β_i be the curve in P_i^{free} that closes $\Pi_i(\underline{\alpha} \oplus \overline{\alpha})$:

- If $d > 2$, as O is contractible (i.e. does not have holes) any path from $\Pi_i(\underline{c})$ to $\Pi_i(\overline{c})$ gives a closed contractible path.
- If $d = 2$, choose β_i to go around B as many times as needed to undo the *winding number* of $\Pi_i(\underline{\alpha} \oplus \overline{\alpha})$ (Fig. 4).

The curve $\beta = (\beta_1 \ldots \beta_n)$ closes $\underline{\alpha} \oplus \overline{\alpha}$ in a contractible way.

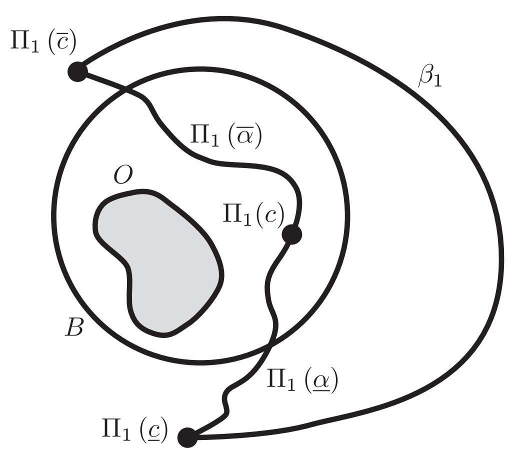


Fig. 4. Completion of the contractible path β_1 in P_1^{free}.

5.3 Intersection of the Contraction with M_c

The contractibility of the constructed path implies the existence of an homotopy H, as in equation (2), that contracts the closed path to c in M^{free}.

Let $H(S)$ be the image of the square $S = [0, 1] \times [0, 1]$ in M by the homotopy map. We are interested in the intersection of the homotopy image with the rigid motions set, $H(S) \cap M_c$. It may seem obvious that an escape path lives within that intersection, as illustrated by the simple case of Fig. 3, but for a more pathological homotopy the path may not exist. We will show that H can always be approximated by a well behaved contraction that yields a nondegenerate intersection.

The construction of the intersection relies on lemma 1, borrowed from differential topology [8].

Lemma 1. *Let M be an m-dimensional manifold and N an n-dimensional manifold, with $m \geq n$. If $f : M \longrightarrow N$ is smooth, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

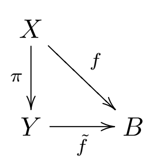
If M is a manifold with boundary and y is also regular for the restriction $f \mid \partial M$, then $f^{-1}(y)$ is a smooth $(m-n)$ manifold with boundary. Furthermore, the boundary $\partial\left(f^{-1}(y)\right)$ is precisely equal to the intersection of $f^{-1}(y)$ with ∂M.

The rest of this section uses lemma 1 to construct the intersection $H(S) \cap M_c$, requiring some special care to satisfy the smoothness requirements.

To simplify later arguments we will change the domain of the homotopy from a square to a disc. We can view the homotopy H as a parametrization of the set $H(S)$, where ∂S is mapped to the contractible curve $\underline{\alpha} \oplus \beta \oplus \overline{\alpha}$ with the inconvenience that three sides of S are mapped to c. Let π be the quotient map that identifies those three sides of the square into one single point q. The map π transforms S into a disc D, whose boundary is a one to one mapping of the contractible curve.

The characteristic properties of the quotient topology [6], expressed in theorem 2, guarantee the existence and uniqueness of a continuous map $\tilde{H} : D \longrightarrow M^{\text{free}}$ that commutes the diagram on Fig. 5. From now on all mentions to the contraction will refer to that quotient induced map $\tilde{H}$.

Theorem 2 (Passing to the Quotient). *Suppose $\pi : X \longrightarrow Y$ is a quotient map, B is a topological space, and $f : X \longrightarrow B$ is any continuous map that is constant on the fibers of π (i.e., if $\pi(p) = \pi(q)$, then $f(p) = f(q)$). Then there exists a unique continuous map $\tilde{f} : Y \longrightarrow B$ such that $f = \tilde{f} \circ \pi$:*



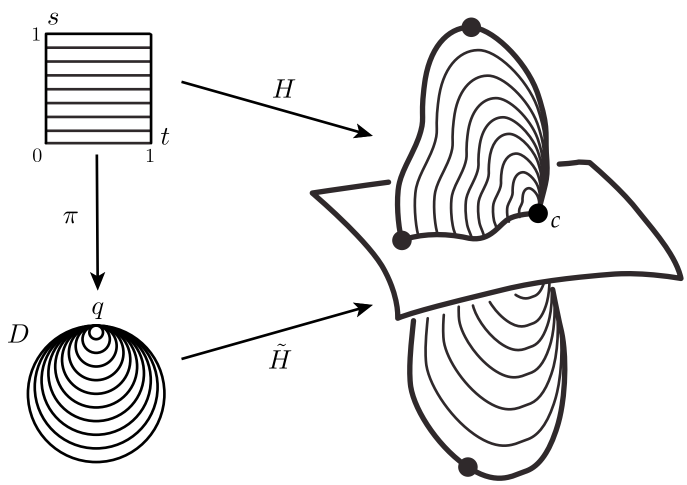


Fig. 5. Construction of the homotopy $\tilde{H}$ induced by the quotient map. See that $\tilde{H}(q) = c$.

Let F be the composition of the contraction $\tilde{H}$ with the distance map r:

$$F : D \xrightarrow{\tilde{H}} M^{\text{free}} \xrightarrow{r} \mathbb{R} \tag{7}$$

If r_c is the distance between the finger points at c, then $F^{-1}(r_c)$ parameterizes the intersection set $\tilde{H}(D) \cap M_c$. $\tilde{H}$ maps the set $F^{-1}(r_c) \cap \partial D$ to the crossings of the contractible curve through M_c. By construction, except for q that maps to c, all points in $F^{-1}(r_c) \cap \partial D$ are mapped by $\tilde{H}$ to noncaging configurations.

Showing that c is also a noncaging configuration – i.e. there is a rigid escape path for c – is equivalent to finding a path from q to any other point in ∂D within $F^{-1}(r_c)$. Lemma 1 allow us to construct the set $F^{-1}(r_c)$ and prove the existence of that path.

If F is smooth and r_c is a regular value, lemma 1 says that $F^{-1}(r_c)$ is a one-dimensional smooth manifold with the set $F^{-1}(r_c) \cap \partial D$ as boundary – a finite union of copies of $\mathcal{S}^1$ entirely in the interior of D and smooth paths that begin and end in ∂D. Since $q \in F^{-1}(r_c)$ and $q \in \partial D$, it follows that q must be connected to another point in ∂D within $F^{-1}(r_c)$. Thus, the connecting curve in D maps to a rigid escape path from q and the theorem is true. All that remains is to see what happens when F is not smooth or r_c is not a regular value.

Smoothness of $\tilde{H}$

To apply lemma 1 to F, $F = r \circ \tilde{H}$ must be smooth, hence the contraction $\tilde{H}$ needs to be smooth. For the contraction to be smooth, the contractible curve $\underline{\alpha} \oplus \beta \oplus \overline{\alpha}$ must also be smooth. We address this issue using theorem 3, Whitney's Approximation Theorem [7], to construct smooth ϵ-approximations of both.

Theorem 3 (Whitney Approximation Theorem). *Let M be a smooth manifold and let $F : M \longrightarrow \mathbb{R}^k$ be a continuous function. Given any positive continuous function $\epsilon : M \longrightarrow \mathbb{R}$, there exists a smooth function $\hat{F} : M \longrightarrow \mathbb{R}^k$ that is ϵ-close to F ($\|F(x) - \hat{F}(x)\| < \epsilon(x)\ \forall x \in M$). If F is smooth on a closed subset $A \subset M$, then $\hat{F}$ can be chosen to be equal to F in A.*

The challenge is to preserve two key properties of $\tilde{H}$ when we construct the approximation:

1. All crossings of M_c through the boundary of $\tilde{H}(D)$, other than c, must remain noncaging.
2. The approximation of $\tilde{H}$ must still live in M^{free}.

The first property was obtained by making the contractible curve go *to infinity* before crossing M_c. Spurious crossings of M_c must be prevented. This is especially awkward near the crossing at c. No matter how small an ϵ we choose, the ϵ-approximation of $\tilde{H}$ could cross again. Similarly, where $\tilde{H}$ makes contact with the obstacle, the ϵ-approximation might violate the second property by leaving the free space M^{free}. The following procedure produces a smooth approximation while avoiding the problems:

1. Replace the contractible curve locally at c by a smooth *patch* in M^{free}, as shown in Fig. 6. This is possible because of our "stricter" definition of free space, ensuring the free space has no thin bits. Even if c is in contact with O^M, there is still freedom to smoothly escape the contact through half of the directions on the tangent space.
2. Apply a similar patch wherever the contractible curve contacts the object. Thanks to the regularity of the free space, proposition 1 guarantees that these contact points are isolated configurations.
3. Apply theorem 3 to approximate the contractible curve by a smooth curve, equal to the original curve on the patches.
4. Define the contraction $\tilde{H}$ as before, but using the smoothed contractible curve.
5. If the contraction makes contact with the object, the approximation could violate the second key property. Repeat the previous strategy of defining smooth patches.
6. Apply theorem 3 once more to approximate the contraction by a smooth one $\hat{H}$ that equals the original contraction on the closed set $\partial D \subset D$ and on any patches, and otherwise lives in the free space $M^{\mathbf{free}}$.

Regularity of r_c

If r_c is a regular value of the now smooth mapping $\hat{F} = r \circ \hat{H}$, lemma 1 says that $F^{-1}(r_c)$ is a one-dimensional smooth manifold. We have seen that

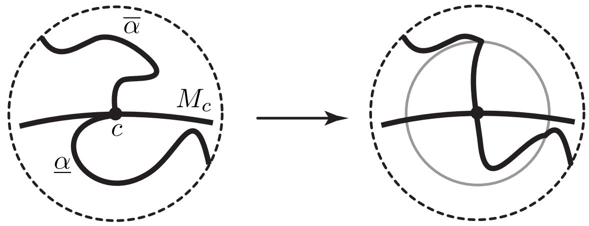


Fig. 6. Smooth patch for replacing the contractible curve in a neighborhood of c and eliminate possible nonsmoothness at c.

consequently there is a smooth escape path within $F^{-1}(r_c)$ that connects q with another point in ∂D, mapped by $\hat{H}$ to a known noncaging configuration. But if r_c is not regular then $F^{-1}(r_c)$ might not be a manifold and the argument fails. This section shows that in such cases the escape path connecting q with a boundary point of $F^{-1}(r_c)$ still exists.

Sard's Theorem characterizes the critical points of a smooth map:

Theorem 4 (Sard's Theorem). *Let $f : U \longrightarrow \mathbb{R}^p$ be a smooth map, with U open in R^n and let C be the set of critical points; that is the set of all $x \in U$ with rank $df_x < p$. Then $f(C) \subset \mathbb{R}^p$ has measure zero.*

The theorem says that for a smooth real valued function, the set of regular values is dense on the image of the function. Consequently, given any critical value z of the smooth function f, there is a monotonic sequence of regular values converging to z, $\{z_n\}_n \longrightarrow z$.

In the specific case of the smooth map $\hat{F} : D \longrightarrow \mathbb{R}$, if $r_c \in \mathbb{R}$ is a critical value, let $\{r_n\}$ be a monotonic sequence of regular values converging to r_c. Let ϵ_n be the positive difference $r_c - r_n$ so that:

$$\{\epsilon_n\}_n \longrightarrow 0 \text{ and } \{r_c - \epsilon_n\}_n \longrightarrow r_c \tag{8}$$

We know that the contractible path goes from $\underline{M}_c$ to $\overline{M}_c$ locally at c. As $\hat{H}$ maps ∂D to the contractible curve, the restriction $\hat{F}\,|_{\partial D}$ is monotonic in a neighborhood of q. Therefore we can expect to find a monotonic sequence of points $\{q_{\epsilon_n}\}_n$ in a neighborhood of q at ∂D that converges to q, such that $q_{\epsilon_n} \in F^{-1}(r_c - \epsilon_n)\ \forall n$.

The sets $F^{-1}(r_c - \epsilon_n)$ are smooth one-dimensional manifolds, because $r_c - \epsilon_n = r_n$ are regular values. Consequently there is a smooth path from q_{ϵ_n} to another point in ∂D within $F^{-1}(r_c - \epsilon_n)$. The sequence of regular values $\{r_c - \epsilon_n\}_n$ induces a sequence of smooth paths that gradually approaches the set $F^{-1}(r_c)$ with $\epsilon_n \longrightarrow 0$, as illustrated in Fig. 7.

Each path of the succession defines a subset $R_n \subset D$ bounded partially by the smooth path and partially by the boundary of D itself. Each region in the sequence is a subset of the next, because two smooth paths of different regular values cannot cross, and the sequence of regular values increases monotonically.

The union of all those regions $R = \bigcup_n R_n$ defines a set in D that, by construction, is partially bounded by ∂D and the set $F^{-1}(r_c)$. The section of that boundary within the set $F^{-1}(r_c)$ provides the desired path. Since $\{q_{\epsilon_n}\}$ converges on q, the *limit path* connects q with another point in ∂D. Thus in Fig. 7 while $F^{-1}(r_c)$ might not be a manifold, and the path obtained might not be smooth, it still exists.

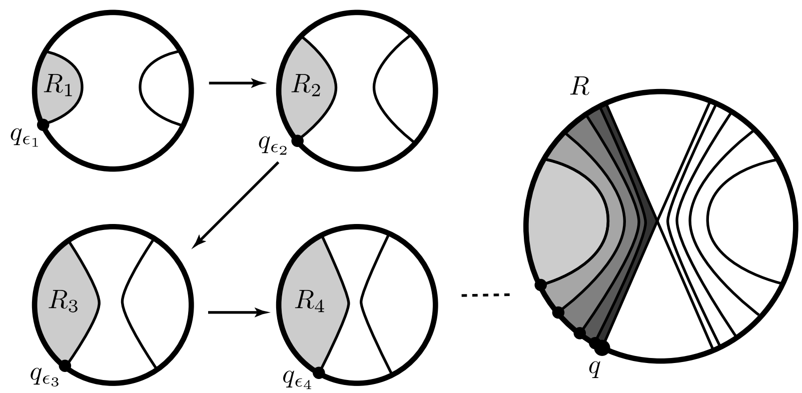


Fig. 7. For every ϵ_n, there is a smooth path in $F^{-1}(r_c - \epsilon_n)$ from q_{ϵ_n}. The sequence of those paths defines in the limit the escape path from q.

5.4 Requirements Revisited

Theorem 1 imposes three requirements on the object O as a subset of $\mathbb{R}^d$: compactness, connectedness and contractibility. The theorem implies their sufficiency, but not their necessity.

- **Compactness:** One of the requirements when building the contractible curve is that all crossings through M_c should be known noncaging configurations, except for c. We obtain this by constructing a ball around the object, which easily addresses the issue for compact objects. However, noncompact objects do not necessarily falsify the theorem.
- **Connectedness**: We used connectedness to build the contractible curve in a systematic way. For nonconnected objects (Fig. 8) it may be impossible to undo the winding number of $\underline{\alpha} \oplus \overline{\alpha}$ outside the ball B, but there may be some other way to construct a suitable contractible curve.
- **Contractibility**: If the object O has holes and $d > 2$, we may again be unable to close the path in a contractible way outside the ball B. Contractibility is not required in the planar case, because:

– If one or two fingers are inside a hole, the object is clearly caged, squeezing caged and stretching caged.
– If none of the fingers are inside a hole, the hole is irrelevant and Theorem 1 applies directly.

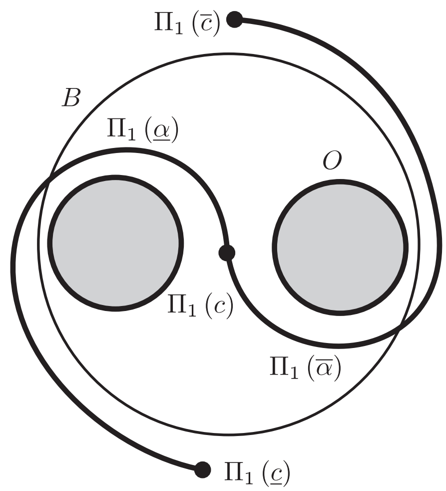


Fig. 8. Impossible to undo the winding number of $\underline{\alpha} \oplus \overline{\alpha}$ outside B, due to the nonconnectedness of O.

5.5 Implications

The squeezing and stretching theorem (theorem 1) gives an interesting characterization of caging configurations: if an object is trapped, it will remain trapped even if we allow the point fingers a partial motion freedom: squeezing in some cases, stretching in others.

This squeezing and stretching result connects caging to immobilization. Suppose that c is a caging configuration, and for example it is the squeezing caging case. By definition, c lies in a compact connected component of $M^{\text{free}} \cap \underline{M}_c$. If the fingers squeeze, even in a *blind* way, the motion is bounded, and the path must inevitably end in an immobilization, at least for frictionless generic shapes. Caging helps to solve the immobilization problem by thickening the zero measure set of immobilizing configurations to regions of positive measure.

The immobilization problem is the problem of finding a manipulator configuration that eliminates the object's freedom of movement, given a geometric description of its shape and location. The formulation of the problem has two main inconveniences:

- The set of solutions to the immobilization problem is a subset of the contact space. Therefore, the set of solutions to the problem, as a subset of the configuration space of the manipulator, has measure zero.
- The theoretical formulation of the problem relies on unrealistic assumptions such as having a *perfect* geometrical model of the object both in terms of shape and location, and the ability to place the manipulator *perfectly* in a specific configuration.

These two properties would seem to make immobilization impractical, since any error would make it impossible to place the manipulator on a set of measure zero.

Nonetheless, manipulators do achieve immobilizing configurations. Feedback of contact sensor data is part of the answer. More importantly, the inherent compliance of the effector mechanism, the servos, and the object can accommodate errors. Even if the robot does not reach the desired configuration, it may reach a nearby immobilizing configuration.

The squeezing and stretching theorem facilitates the process of achieving an immobilization by providing an initial condition and a blind strategy for achieving an immobilization.

6 Conclusions and Future Work

Our main result is that any caging of a compact connected contractible object in $\mathbb{R}^d$ by two points is either a squeezing caging, a stretching caging, or both. This generalizes the result of Vahedi and van der Stappen [18] which assumes a polygonal object in the plane. Vahedi and van der Stappen developed the squeezing and stretching caging idea to compute the caging configurations for planar polygons. The generalization suggests that the squeezing and stretching caging idea is a fundamental attribute giving structure to the configuration space of two-fingered manipulators.

Section 5.5 shows that Theorem 1 addresses the immobilization problem. From any caging configuration there is a *blind policy* to immobilize the object. The result says that caging regions are partially bounded by immobilizations, and that there is always a policy to reach them. Therefore, caging thickens the zero measure set of immobilizing configurations to regions of positive measure.

Does the result generalize to n fingers? One approach is to define an n finger manipulator with a one-dimensional shape space [2], but there are other possible generalizations. One advantage of the topological approach is that the entire proof is independent of dimension, and should yield a natural generalization of the squeezing-stretching concept to the case of n fingers.

A. Contractible Paths

Proposition 3 (Characterization of contractible paths). *A closed path α at c is contractible in M^{free} if and only if $\Pi_i(\alpha)$ describes a contractible path in P_i^{free} $\forall i$.*

Proof. The result is a consequence of M being the cartesian product of the configuration space of each finger point, $M = \otimes_{i=1}^{n} P_i$. Let's prove both implications:

[$\Rightarrow$] Suppose α is contractible in M^{free}. Let $H(t,s)$ be the corresponding path homotopy. The natural projections of α are the closed curves $\Pi_i(\alpha) \subset P_i^{\text{free}}$. We need to show that α_i is contractible in P_i^{free} $\forall i$.

Consider then the natural projections of the homotopy $H(s,t)$:

$$\begin{array}{rcccc} H_i : [0,1]\times[0,1] & \xrightarrow{H} & M^{\text{free}} & \xrightarrow{\Pi_i} & P_i^{\text{free}} \\ (t,s) & \rightarrow & H(t,s) & \rightarrow & \Pi_i\left(H(t,s)\right) \end{array} \tag{9}$$

Each H_i is a continuous map because it is a composition of continuous maps ($H_i = \Pi_i \circ H$). Each H_i is a homotopy of paths because:

$$\begin{array}{ll} H_i(t,0) = \Pi_i\left(H(t,0)\right) = \Pi_i\left(\alpha(t)\right) = \alpha_i(t) & \forall t \\ H_i(t,1) = \Pi_i\left(H(t,1)\right) = \Pi_i(c) & \forall t \\ H_i(0,s) = H_i(1,s) = \Pi_i\left(H(0,s)\right) = \Pi_i(c) & \forall s \end{array} \tag{10}$$

We conclude that each α_i is contractible.

[$\Leftarrow$] Suppose that each natural projection $\alpha_i = \Pi_i(\alpha)$ is contractible in its corresponding space P_i^{free}. Let $H_i(t,s) \subset P_i^{\text{free}}$ be the corresponding path homotopy.

Consider the path in M, $\alpha = (\alpha_1 \ldots \alpha_n)$. Note that by construction $\alpha \subset M^{\text{free}}$:

> Suppose $\exists\ t \mid \alpha(t) \notin M^{\text{free}} \overset{eq.(1)}{\Longleftrightarrow} \alpha(t) \in O^M \iff \exists\ i \mid \alpha(t) \in O_i^M$. By definition of O_i^M that happens iff $\alpha_i(t) \in O$. However this contradicts $\alpha_i(t)$ being defined on P_i^{free} by hypothesis. Therefore we conclude that $\alpha \subset M^{\text{free}}$ and is well defined.

Consider now the map $H = (H_1(t,s) \ldots H_n(t,s))$. same way we proved it for α we know that $H(t,s) \subset M^{\text{free}}$, and therefore is well defined. It suffices to check that:

$$\begin{array}{ll} H(t,0) = (\alpha_1(t) \ldots \alpha_n(t)) = \alpha(t) & \forall t \\ H(t,1) = (\alpha_1(0) \ldots \alpha_n(0)) = p & \forall t \\ H(0,s) = H(1,s) = (\alpha_1(0) \ldots \alpha_n(0)) = p & \forall t \end{array} \tag{11}$$

to conclude that H is a path homotopy for α and therefore, α is contractible.

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